Solutions to Selected Exercises in

High-Dimensional Probability (2nd ed.)

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November 18, 2026

0 Appetizer: Using Probability to Cover a Set

0.1 (Two variance formulas)

(a) Recall that the variance of a random variable X satisfies $\mathrm{Var}(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$. Let us prove a higher-dimensional version of this identity. Check that any random vector Z in $\mathbb{R}^n$ satisfies

$$\mathbb{E}\|Z - \mathbb{E}Z\|_2^2 = \mathbb{E}\|Z\|_2^2 - \|\mathbb{E}Z\|_2^2.$$

(b) Let Z be a random vector in $\mathbb{R}^n$, and Z' be an independent copy of Z, i.e. a random vector independent of Z and with the same distribution as Z. Check that

$$\mathbb{E}\|Z - \mathbb{E}Z\|_2^2 = \frac{1}{2}\mathbb{E}\|Z - Z'\|_2^2.$$

Solution:

(a) Trivial.

(b)

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left\|Z - Z'\right\|_2^2 \\
&= \frac{1}{2}\sum_{i=1}^{n}\mathbb{E}(Z_i - Z_i')^2
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\sum_{i=1}^{n}(\mathbb{E}Z_i^2 - 2\mathbb{E}Z_i Z_i' + \mathbb{E}Z_i'^2) \\
&= \frac{1}{2}\sum_{i=1}^{n}(\mathbb{E}Z_i^2 - 2\mathbb{E}Z_i\mathbb{E}Z_i' + \mathbb{E}Z_i'^2) \quad (Z \text{ and } Z' \text{ are independent.}) \\
&= \sum_{i=1}^{n}(\mathbb{E}Z_i^2 - (\mathbb{E}Z_i)^2) \quad (Z \text{ and } Z' \text{ are identically distributed.}) \\
&= \mathbb{E}\left\|Z - Z'\right\|_2^2.
\end{aligned}
$$

0.2 (Expectation minimizes the mean squared error) The variance of a random variable X has the following extremal property:

$$\mathrm{Var}(X) = \mathbb{E}\min_{a\in\mathbb{R}}(X-a)^2.$$

Let us prove a more general, high-dimensional version of this fact. Check that a random vector Z in $\mathbb{R}^n$ with finite $\mathbb{E}\|Z\|_2^2$ satisfies

$$\mathbb{E}\|Z - \mathbb{E}Z\|_2^2 = \min_{a\in\mathbb{R}^n}\mathbb{E}\|Z-a\|_2^2.$$

Solution: We have

$$
\begin{aligned}
\mathbb{E}\left\|Z-a\right\|_2^2 &= \mathbb{E}\left[(Z-a)^\top(Z-a)\right] \\
&= \mathbb{E}\left[Z^\top Z - 2a^\top Z + a^\top a\right] \\
&= a^\top a - 2a^\top\mathbb{E}[Z] + \mathbb{E}[Z^\top Z].
\end{aligned}
$$

This is a quadratic function of a. To find the minimum with respect to a, we can complete the square. Let $\mu = \mathbb{E}[Z]$:

$$\mathbb{E}\left\|Z-a\right\|_2^2 = (a^\top a - 2a^\top\mu + \mu^\top\mu) - \mu^\top\mu + \mathbb{E}[Z^\top Z].$$

This simplifies to:

$$\mathbb{E}\left\|Z-a\right\|_2^2 = \|a-\mu\|_2^2 + \left(\mathbb{E}[Z^\top Z] - \mu^\top\mu\right).$$

The minimum is attained when $a = \mu$.

0.4 (Balancing vectors) Let $x_1, \ldots, x_k$ be vectors in $\mathbb{R}^n$ that lie within the unit Euclidean ball centered at the origin.

(a) Prove that it is possible to assign a sign $\pm$ to each vector such that the sum $\pm x_1 \pm x_2 \pm \cdots \pm x_k$ lies within a Euclidean ball of radius $\sqrt{k}$ centered at the origin.

(b) Explain why the value $\sqrt{k}$ cannot be reduced.

Solution:

(a) Let W_i, $i = 1, ..., k$ be k i.i.d. Rademacher random variables. Then we have

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{i=1}^{k} W_i x_i\right\|_2^2 \\
&= \sum_{i=1}^{k} \mathbb{E}\left\|W_i x_i\right\|_2^2 \quad \text{(By Exercise 0.3.)} \\
&\leq k \quad \text{(Since } W_i x_i \text{ lies in the unit ball.)}
\end{aligned}$$

There must exist at least one specific outcome $\{w_1, ..., w_k\} \in \{-1, 1\}^k$ such that $\left\|\sum_{i=1}^{k} w_i x_i\right\|_2 \leq \sqrt{k}$.

(b) Let $\{\sigma_1, ..., \sigma_k\} \in \{-1, 1\}^k$ be an arbitrarily fixed sequence of signs. Let $X_1, ..., X_k$ be k random vectors uniformly distributed on the unit Euclidean sphere S^{n-1}. Then we have

$$\begin{aligned}
&\mathbb{E}\left\|\sum_{i=1}^{k} \sigma_i X_i\right\|_2^2 \\
&= \sum_{i=1}^{k} \mathbb{E}\left\|\sigma_i X_i\right\|_2^2 \quad \text{(By Exercise 0.3.)} \\
&= k \quad \text{(Since } \sigma_i X_i \text{ lies on the unit sphere.)}
\end{aligned}$$

Therefore, for any fixed sequance of signs $\{\sigma_1, ..., \sigma_k\} \in \{-1, 1\}^k$ there must exist at least on outcome $\{x_1, ..., x_k\}$ on S^{n-1} such that $\left\|\sum_{i=1}^{k} \sigma_i x_i\right\|_2 \geq \sqrt{k}$.

0.5 (Approximate Carathéodory is asymptotically tight) Demonstrate by example that the bound in Theorem 0.0.2 is almost tight. Specifically, for every n find a set $T \subset \mathbb{R}^n$ such that

for any convex combination $\sum_{j=1}^k \lambda_j x_j$ of any k points $x_1, \ldots, x_k \in T$, one has

$$\left\| x - \sum_{j=1}^k \lambda_j x_j \right\|_2 \geq \sqrt{\frac{1}{k} - \frac{1}{n}}.$$

Let $n \to \infty$ while keeping k fixed to see that Theorem 0.0.2 is asymptotically tight in high dimensions.

Solutions: For a given n, let $T = \{v_1, \ldots, v_n\} \subset \mathbb{R}^n$, where $v_i = e_i - \frac{1}{n}\mathbf{1}$ for $i = 1, \ldots, n$, with e_i being the i-th standard basis vector and $\mathbf{1}$ the vector of all ones. We have,

$$\begin{aligned} v_i^\top v_j &= -\frac{1}{n}, \ i \neq j, \\ \|v_i\|_2^2 &= 1 - \frac{1}{n}. \end{aligned}$$

Let the target point $x = \mathbf{0} \in \mathrm{conv}(T)$, for any convex combination $\sum_{i=1}^k \lambda_i v_i$, we have

$$\begin{aligned} &\left\| \mathbf{0} - \sum_{i=1}^k \lambda_i v_i \right\|_2^2 \\ &= \sum_{i=1}^k \lambda_i^2 \|v_i\|_2^2 + 2 \sum_{1 \leq j < l \leq k} \lambda_j \lambda_l v_j^\top v_l \\ &= \frac{n-1}{n} \sum_{i=1}^k \lambda_i^2 - \frac{2}{n} \sum_{1 \leq j < l \leq k} \lambda_j \lambda_l \\ &= \sum_{i=1}^k \lambda_i^2 - \frac{1}{n} \left(\sum_{i=1}^k \lambda_i \right)^2 \\ &\geq \frac{1}{k} \left(\sum_{i=1}^k \lambda_i \right)^2 - \frac{1}{n} \left(\sum_{i=1}^k \lambda_i \right)^2 \quad \text{(Cauchy's inequality.)} \\ &= \frac{1}{k} - \frac{1}{n}. \quad (\sum_{i=1}^k \lambda_i = 1.) \end{aligned}$$

Therefore, we complete the proof.

0.6 (Bounds on binomial coefficients) Prove the inequalities

$$\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \sum_{j=0}^k \binom{n}{j} \leq \left(\frac{en}{k}\right)^k$$

for any integers $1 \le k \le n$.

Solutions: We prove $(n/k)^k \le \binom{n}{k}$ and $\sum_{j=0}^{k} \binom{n}{j} \le (en/k)^k$. Since

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdots \frac{n-k+1}{1},$$

the first inequality obviously holds by noting $n/k \le (n-j)/(k-j)$ for $1 \le j \le k-1$. We observe

$$\left(\frac{en}{k}\right)^k = e^k \left(\frac{n}{k}\right)^k \ge \left(1 + k + \frac{k^2}{2!} + \cdots + \frac{k^k}{k!}\right) \left(\frac{n}{k}\right)^k.$$

Note that for $j = 0, \ldots, k$,

$$\frac{k^j}{j!} \left(\frac{n}{k}\right)^k = \frac{n^j}{j!} \cdot \frac{n^{k-j}}{k^{k-j}} \ge \frac{n^j}{j!} \ge \frac{n(n-1)\cdots(n-j+1)}{j!} = \binom{n}{j}.$$

Thus we have

$$\left(\frac{en}{k}\right)^k = e^k \left(\frac{n}{k}\right)^k \ge \sum_{j=0}^{k} \frac{k^j}{j!} \left(\frac{n}{k}\right)^k \ge \sum_{j=0}^{k} \binom{n}{j}.$$

0.7 (Thin shell phenomenon) Let us prove a counterintuitive fact that most of the volume of the high-dimensional ball lies near the surface. Consider the points inside the unit Euclidean ball of $\mathbb{R}^n$ that lie within distance $5/n$ from the surface of the ball, see Figure 0.3. Prove that such points make up over 99% of the volume of the ball.

Solutions: WLOG, we assume $n \ge 6$. Let $B^n(r)$ denote the Euclidean ball in $\mathbb{R}^n$ with radius r.

$$\frac{\mathrm{vol}(B^n(1-5/n))}{\mathrm{vol}(B^n(1))} = \left(1 - \frac{5}{n}\right)^n$$

Since the sequence $\{(1-5/n)^n\}$ monotonically converges to $1/e^5 < 0.01$, we complete the proof.

0.8 (Thin shell phenomenon, continued) Let X be a random vector that is uniformly distributed in the Euclidean unit ball of $\mathbb{R}^n$. Prove that

$$\mathbb{E}\|X\|_2 = \frac{n}{n+1}.$$

Solutions: We have

$$\mathbb{P}(\|X\|_2 \leq x) = \frac{\mathrm{vol}(B^n(x))}{\mathrm{vol}(B^n(1))} = x^n.$$

Since $\|X\|_2$ is in $[0,1]$ almost surely, we have

$$\mathbb{E}\,\|X\|_2 = \int_0^1 P(X > t)dt = \int_0^1 1 - x^n dx = \frac{n}{n+1}.$$

0.9 (Carl-Pajor theorem) Let's improve Theorem 0.0.4 by replacing N with N/n. Let P be a polytope with $N \geq n$ vertices, which is contained in the unit Euclidean ball of $\mathbb{R}^n$, denoted by B. Prove that

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \leq \left(C\sqrt{\frac{\log(eN/n)}{n}}\right)^n$$

where $C > 0$ is an absolute constant.

Solution: To begin with, we improve the bound on $|\mathcal{N}|$ in Corollary 0.0.3 using the following fact: the number of ways to choose an unordered subset of k elements from an N-element set, with possible repetitions, is $\binom{N+k-1}{k}$. The volume of P is bounded by the total volume of the balls that cover P, thus

$$\mathrm{Vol}(P) \leq \binom{N+k-1}{k}(1/\sqrt{k})^n \,\mathrm{Vol}(B),$$

Using the result in Exercise 0.6, we obtain

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \leq \frac{(e(N+k-1)/k)^k}{k^{n/2}}.$$

Let us choose $k = n/\log(eN/n)$. Substituting this value yields

$$\frac{\mathrm{Vol}(P)}{\mathrm{Vol}(B)} \leq \left(\frac{e(N+k-1)}{k}\right)^k \left(\sqrt{\frac{\log(eN/n)}{n}}\right)^n.$$

Now it suffices to show

$$\left(\frac{e(N+k-1)}{k}\right)^{k/n} \leq C.$$

Indeed,

$$\left(\frac{e(N+k-1)}{k}\right)^{k/n}$$
$$\leq \left(\frac{e(N+k)}{k}\right)^{k/n}$$

$$\leq \left(\frac{eN\log(eN/n)}{n} + e\right)^{1/\log(eN/n)}$$
$$\leq \left(Le^L + e\right)^{1/L}. \quad (L := \log(eN/n).)$$

Since $L \geq 1$ by the assumption $N \geq n$ and $\left(Le^L + e\right)^{1/L} \to e$ as $L \to \infty$, we finish the proof.

1 A Quick Refresher on Analysis and Probability

1.5 (Expressing a cube as a convex hull of its vertices) It seems almost obvious that the cube is the convex hull of its vertices:

$$[-1,1]^n = \operatorname{conv}\left(\{-1,1\}^n\right).$$

Prove this by expressing any point in the cube as a convex combination of the vertices.

Solution: For each $v = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1,1\}^n$, define

$$\lambda_v = \prod_{i=1}^{n} \frac{1+\varepsilon_i x_i}{2}.$$

Obviously $\lambda_v \geq 0$ for any $v \in \{-1,1\}^n$ We will verify that $\sum_{v\in\{-1,1\}^n} \lambda_v = 1$ and $x = \sum_{v\in\{-1,1\}^n} \lambda_v v$. First,

$$\begin{aligned}
\sum_{v\in V} \lambda_v &= \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n} \prod_{i=1}^{n} \frac{1+\varepsilon_i x_i}{2} \\
&= \frac{1}{2^n} \prod_{i=1}^{n} \sum_{\varepsilon_i\in\{-1,1\}} (1+\varepsilon_i x_i) \\
&= \frac{1}{2^n} \prod_{i=1}^{n} \left((1+x_i) + (1-x_i)\right) \\
&= \frac{1}{2^n} \prod_{i=1}^{n} 2 \\
&= 1.
\end{aligned}$$

We also have for any $j \in \{1, \ldots, n\}$,

$$\left(\sum_{v\in V} \lambda_v v\right)_j$$

$$
\begin{aligned}
&= \sum_{v \in V} \lambda_v v_j \\
&= \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n} \varepsilon_j \prod_{i=1}^{n} \frac{1+\varepsilon_i x_i}{2} \\
&= \frac{1}{2^n} \left(\sum_{\varepsilon_j \in \{-1,1\}} \varepsilon_j (1+\varepsilon_j x_j) \right) \left(\prod_{i \neq j} \sum_{\varepsilon_i \in \{-1,1\}} (1+\varepsilon_i x_i) \right) \\
&= \frac{1}{2^n} \left((1+x_j) - (1-x_j) \right) \left(\prod_{i \neq j} 2 \right) \\
&= \frac{1}{2^n} (2x_j)(2^{n-1}) \\
&= x_j.
\end{aligned}
$$

We complete the proof.

1.6 (Expressing a cross-polytope as a convex hull of its vertices) Check that the unit ball corresponding to the ℓ^1 norm in $\mathbb{R}^n$ is the *absolute convex hull* of the standard basis $e_1, \ldots, e_n$ in $\mathbb{R}^n$, that is

$$B_1^n = \operatorname{conv}\left(\{\pm e_1, \ldots, \pm e_n\}\right).$$

Write down a formula that expresses any point $x \in B_1^n$ as a convex combination of the vectors $\pm e_1, \ldots, \pm e_n$.

Solution: For $i \in \{1, \ldots, n\}$, define

$$
\begin{aligned}
u_i &= \mathbb{1}\{x_i > 0\} x_i + (1 - \|x\|_1)/2n \\
v_i &= \mathbb{1}\{x_i < 0\} x_i + (1 - \|x\|_1)/2n.
\end{aligned}
$$

Obviously $u_i \geq 0$, $v_i \geq 0$. We now verify that these coefficients sum to one:

$$
\begin{aligned}
\sum_{i=1}^{n} (u_i + v_i) &= \sum_{i=1}^{n} \left(|x_i| + \frac{1 - \|x\|_1}{n} \right) \\
&= \|x\|_1 + 1 - \|x\|_1 \\
&= 1.
\end{aligned}
$$

Also, for each $i \in \{1, \ldots, n\}$,

$$\left[\sum_i (u_i e_i + v_i(-e_i))\right]_i = u_i - v_i = x_i.$$

Thus $x = \sum_i (u_i e_i + v_i(-e_i))$. We complete the proof.

1.7 (Random graphs with random number of vertices) Suppose that in Example 1.4.2, the number n of freshmen who arrive on campus is a random variable that has Poisson distribution with mean λ. As before, each pair of students becomes friends with probability p independent of all other pairs. Show that if $p \geq 2\ln(\lambda)/\lambda$ then there are no friendless students with probability at least $1 - 1/\lambda$.

Solution: Let B denote the event that there exists at least one friendless student. From Example 1.4.2, $P(B \mid X = x) \leq x(1-p)^{x-1}$. Therefore, we have

$$\begin{aligned}
P(B) &= \sum_{x=0}^{\infty} P(B \mid X = x) P(X = x) \\
&\leq \sum_{x=0}^{\infty} x(1-p)^{x-1} \frac{e^{-\lambda}\lambda^x}{x!} \\
&= \sum_{x=1}^{\infty} \lambda e^{-\lambda} \frac{(\lambda(1-p))^{x-1}}{(x-1)!} \\
&= \lambda \exp\left(-\lambda + \lambda(1-p)\right) \\
&= \lambda \exp\left(-\lambda p\right) \\
&\leq \frac{1}{\lambda}.
\end{aligned}$$

We complete the proof.

1.8 (Independent sets in random graphs) Call a group of people *independent* if no two members are friends. Suppose $n \geq 7$ students enroll in a class on high-dimensional probability, with each pair becoming friends independently with probability $1/2$. Show that, with probability at least $1 - 1/n$, this class has no independent subsets of more than $2\log_2 n$ students.

Solution: Let $m = \lfloor 2\log_2 n \rfloor + 1$. Our goal is to show that the probability of an independent set of size m existing is at most $1/n$.

Let S be a specific subset of m students. The probability of S being an independent set is

$$P(S \text{ is independent}) = \left(\frac{1}{2}\right)^{\binom{m}{2}}.$$

Let A be the event that there exists an independent set of size m. By the union bound over all $\binom{n}{m}$ subsets of size m, we have:

$$P(A) \leq \binom{n}{m}\left(\frac{1}{2}\right)^{\binom{m}{2}}.$$

Using the standard inequality $\binom{n}{m} \leq \left(\frac{en}{m}\right)^m$, we have:

$$P(A) \leq \left(\frac{en}{m}\right)^m 2^{-\frac{m(m-1)}{2}}.$$

We use $m > 2\log_2 n$ to bound the exponential term:

$$\frac{m(m-1)}{2} = (m-1)\frac{m}{2} > (m-1)\log_2 n.$$

This gives us $2^{-\frac{m(m-1)}{2}} \leq n^{-(m-1)}$. Substituting this back into our bound for $P(A)$ yields:

$$P(A) \leq \left(\frac{en}{m}\right)^m n^{-(m-1)} = \frac{e^m n^m}{m^m}\frac{1}{n^{m-1}} = \frac{1}{n}\cdot n^2\left(\frac{e}{m}\right)^m.$$

The proof now reduces to showing that $n^2(m/e)^m \leq 1$ for $n \geq 7$. Taking the logarithm, we need to establish:

$$2\ln n \leq m(\ln m - 1).$$

We know $m > 2\log_2 n = 2\ln n/\ln 2$. Substituting this lower bound for m, it suffices to prove:

$$2\ln n \leq \frac{2\ln n}{\ln 2}\left(\ln(2\log_2 n) - 1\right).$$

Or equivalently,

$$\ln 2 \leq \ln(2\log_2 n) - 1.$$

Rearranging gives $1 \leq \ln(\log_2 n)$. This final inequality holds if and only if $\log_2 n \geq e$, which is equivalent to $n \geq 2^e \approx 6.58$. We finish the proof.

1.10 (Sparse random graphs have isolated vertices) Let us prove a converse to Exercise 1.9. Fix any $\varepsilon > 0$ and assume that

$$p_n < \frac{(1-\varepsilon)\ln n}{n} \quad \text{for every } n \in \mathbb{N}.$$

Then there exists at least one friendless student with probability that converges to 1 as $n \to \infty$. You will prove this result using the so-called *second moment method*:

(a) Denote the number of friendless students by S_n and express it as $S_n = X_1 + \ldots + X_n$ where X_i is the indicator of the event that student i is friendless. Show that

$$\mu_n = \mathbb{E}S_n \to \infty.$$

Thus the expected number of friendless students is large. But this does not automatically imply that there exists even one friendless student with high probability! (Why?)

(b) Compute the second moment $\mathbb{E}S_n^2$ by expanding the square. Conclude that

$$\frac{\mathrm{Var}(S_n)}{\mu_n^2} \to 0.$$

(c) Use Chebyshev inequality to complete the proof.

Solution:

(a) For any student i, we have

$$\mathbb{E}X_i = \mathbb{P}(X_i = 1) = (1-p_n)^{n-1}.$$

Thus, the expected number of friendless students is

$$\mu_n = n(1-p_n)^{n-1}.$$

Our goal is to show that $\mu_n \to \infty$ as $n \to \infty$. We are given that $p_n < (1-\varepsilon)\log n/n$ for some fixed $\varepsilon > 0$. This implies that $p_n \to 0$ as $n \to \infty$. We can rewrite μ_n as

$$\mu_n = n\exp\big((n-1)\ln(1-p_n)\big).$$

For small x, the Taylor series for $\ln(1-x)$ is $-x - \frac{x^2}{2} - O(x^3)$. Since $p_n \to 0$, we can analyze the exponent as follows:

$$\begin{aligned}
&(n-1)\ln(1-p_n) \\
&= (n-1)\left(-p_n - O(p_n^2)\right) \\
&= -np_n + p_n - O(np_n^2).
\end{aligned}$$

Let's analyze the terms in the exponent. From the given condition,

$$-np_n > -(1-\varepsilon)\ln n.$$

And the other terms vanish as $n \to \infty$. Therefore, the exponent can be bounded below:

$$(n-1)\ln(1-p_n) > -(1-\varepsilon)\ln n + o(1).$$

Substituting this back into the expression for μ_n, we get

$$\mu_n > n\exp\left(-(1-\varepsilon)\ln n + o(1)\right) = n \cdot n^{-(1-\varepsilon)} \cdot e^{o(1)} = n^{\varepsilon} e^{o(1)}.$$

Since $\varepsilon > 0$ and $e^{o(1)} \to 1$ as $n \to \infty$, we conclude that $\mu_n \to \infty$.

A large expectation $\mathbb{E}S_n$ does not imply that $\mathbb{P}(S_n > 0)$ is high. For example, consider a random variable Y_n such that $\mathbb{P}(Y_n = n^2) = 1/n$ and $\mathbb{P}(Y_n = 0) = 1 - 1/n$. Then $\mathbb{E}Y_n = n \to \infty$, but the variable is almost surely zero, as $\mathbb{P}(Y_n = 0) \to 1$.

(b) Expanding $\mathbb{E}S_n^2$ yields

$$\mathbb{E}S_n^2 = \sum_{i=1}^{n} \mathbb{E}X_i^2 + 2 \sum_{1 \le i < j \le n} \mathbb{E}X_i X_j.$$

We have for any $i \in \{1, ..., n\}$, $\mathbb{E}X_i^2 = \mathbb{E}X_i$, and for any $1 \le i < j \le n$,

$$\mathbb{E}X_i X_j = P(X_i = X_j = 1) = (1-p_n)^{2n-3} = (\mu_n/n)^2/(1-p_n).$$

We analyze the ratio $\mathrm{Var}(S_n)/\mu_n^2$:

$$\frac{\mathrm{Var}(S_n)}{\mu_n^2} = \frac{\mathbb{E}S_n^2 - (\mathbb{E}S_n)^2}{\mu_n^2}$$

$$
\begin{aligned}
&= \frac{\mu_n + n(n-1)(\mu_n/n)^2/(1-p_n) - \mu_n^2}{\mu_n^2} \\
&= \frac{1}{\mu_n} + \frac{n(n-1)}{n^2(1-p_n)} - 1 \\
&\to 0.
\end{aligned}
$$

We finish the proof.

(c) By Chebyshev's inequality, we can obtain

$$
\mathbb{P}\left(S_n < \frac{\mu_n}{2}\right) \leq \mathbb{P}\left(|S_n - \mu_n| \geq \mu_n/2\right) \leq \frac{\mathrm{Var}(S_n)}{\mu_n^2/4}.
$$

We finish the proof by noting that $\mathrm{Var}(S_n)/\left(\mu_n^2/4\right) \to 0$ as $n \to \infty$.

1.11 (Monotonicity of the L^p norm)

(a) Let X be a random variable. Show that $\|X\|_{L^p}$ is an increasing function in p:

$$
\|X\|_{L^p} \leq \|X\|_{L^q} \quad \text{for any } 0 \leq p \leq q \leq \infty.
$$

(b) Demonstrate that the inequality in part (a) cannot be reversed: for any $0 \leq p < q \leq \infty$, find an example of a random variable X with $\|X\|_{L^p} < \infty$ and $\|X\|_{L^q} = \infty$.

Solution: (a) We omit the case that $p = 0$ since the L_0 norm is not a well-defined norm. For any $0 < p < q < \infty$, the function $f(x) = x^{q/p}$ is convex. We have

$$
\begin{aligned}
&\|X\|_{L^q} \\
&= (\mathbb{E}|X|^q)^{1/q} \\
&= (\mathbb{E}f(|X|^p))^{1/q} \\
&\geq f(\mathbb{E}|X|^p)^{1/q} \quad \text{(Jensen's inequality.)} \\
&= (\mathbb{E}|X|^p)^{1/p} \\
&= \|X\|_{L^p}.
\end{aligned}
$$

Now we verify the corner case $q = \infty$. It is straightforward to see

$$
\|X\|_{L^\infty} = \mathrm{esssup}|X| = (\mathrm{esssup}|X|^p)^{1/p} \geq (\mathbb{E}|X|^p)^{1/p} = \|X\|_{L^p}.
$$

We finish the proof.

(b) For any $0 < p < q \le \infty$, consider a random variable X such that $P(X = n^{(1-\varepsilon)/p}) = 6/\pi^2 n^2$, for $n = 1, 2, \ldots$, where $\varepsilon > 0$ is a constant such that $q(1-\varepsilon)/p > 1$. Then we have $\mathbb{E}|X|^p < \infty$ but $\mathbb{E}|X|^q = \infty$.

1.12 (Interpolation between L^1 and L^∞) We know the L^p norm of any random variable X is bounded by the L^∞ norm. We can get an even better bound if we also know that the L^1 norm of X is small. Show that

$$\|X\|_{L^p} \le \|X\|_{L^1}^{\frac{1}{p}} \|X\|_{L^\infty}^{1-\frac{1}{p}} \quad \text{for any } 1 < p < \infty.$$

Solution: The desired inequality is equivalent to

$$\mathbb{E}|X|^p \le \mathbb{E}|X| \left(\operatorname{esssup} |X|\right)^{p-1}.$$

The case $\operatorname{esssup} |X| = 0$ is trivial. Otherwise, the inequality can be rewritten as

$$\mathbb{E}\left[|X| \cdot \left(\frac{|X|}{\operatorname{esssup} |X|}\right)^{p-1}\right] \le \mathbb{E}|X|,$$

which obviously holds since $|X| / \operatorname{esssup} |X| \le 1$ almost surely.

1.13 (Expectation of a maximum) Let $X_1, \ldots, X_n$ be nonnegative random variables.

(a) Prove that

$$\max_{i \le n} \mathbb{E}X_i \le \mathbb{E} \max_{i \le n} X_i \le n \cdot \max_{i \le n} \mathbb{E}X_i.$$

(b) Demonstrate that both inequalities in part (a) may be optimal. Specifically, find random variables $X_1, \ldots, X_n$ satisfying $\max_i \mathbb{E}X_i = \mathbb{E} \max_i X_i > 0$ and random variables Y_i satisfying $\mathbb{E} \max_i Y_i = n \cdot \max_i \mathbb{E}Y_i > 0$.

(c) Demonstrate that the upper bound in part (a) may be approximately optimal even for independent random variables. Specifically, find independent random variables $X_1, \ldots, X_n$ satisfying $\mathbb{E} \max_i X_i > cn \cdot \max_i \mathbb{E}X_i$, where $c > 0$ is an absolute constant.

Solution: (a) The first inequality obviously holds as for any $i \le n$, $\mathbb{E}X_i \le \mathbb{E} \max_{i \le n} X_i$. To

prove the second inequality, we have

$$\mathbb{E}\max_{i\le n} X_i \le \mathbb{E}\sum_{i\le n} X_i = \sum_{i\le n}\mathbb{E}X_i \le n\max_{i\le n}\mathbb{E}X_i.$$

(b) To show the first inequality is sharp, simply let X be a non-negative random variable with $\mathbb{E}X > 0$, and set $X_i = X$ for all $i = 1,\dots,n$. To show the second inequality is sharp, we let $U \sim \text{Unif}[0,1]$, and $Y_i = \mathbb{1}\{U \in [(i-1)/n, i/n]\}$. Then for any $i \in \{1,\dots,n\}$, $\mathbb{E}Y_i = 1/n$ but $\mathbb{E}\max_{i\le n} Y_i = 1$.

(c) Let $X_1,\dots,X_n$ be independent random variables defined as follows: for each $i \in \{1,\dots,n\}$,

$$X_i = \begin{cases} i & \text{with probability } 1/i \\ 0 & \text{with probability } 1-1/i. \end{cases}$$

The expectation is $\mathbb{E}X_i = 1$ for all i. To lower bound the expectation of $\max_{i\le n} X_i$, we have

$$\mathbb{P}\left(\max_{i\le n} X_i \le \frac{n}{2}\right) = \left(1-\frac{1}{n/2+1}\right)\cdots\left(1-\frac{1}{n}\right) = 1/2.$$

Therefore,

$$\mathbb{E}\max_{i\le n} X_i \ge \frac{n}{4} = \frac{n}{4}\max_{i\le n}\mathbb{E}X_i.$$

1.14 Let $X_1,\dots,X_n$ be nonnegative random variables. Prove that for any $1 \le p < \infty$, we have

$$\left(\sum_{i=1}^n (\mathbb{E}X_i)^p\right)^{1/p} \le \mathbb{E}\left(\sum_{i=1}^n X_i^p\right)^{1/p} \le \left(\sum_{i=1}^n \mathbb{E}\left(X_i^p\right)\right)^{1/p}.$$

Solution: When $p \ge 1$, the function $f(x) = x^p$ is convex. By Jensen's inequality,

$$f\left(\mathbb{E}\left(\sum_{i=1}^n X_i^p\right)^{1/p}\right) \le \mathbb{E}f\left(\left(\sum_{i=1}^n X_i^p\right)^{1/p}\right) = \sum_{i=1}^n \mathbb{E}X_i^p.$$

Taking the p-th root of the left and right sides yields the first inequality. For the second inequality, the function $g(x) = x^{1/p}$ is concave. Again, we apply Jensen's inequality,

$$\mathbb{E}g\left(\sum_{i=1}^n X_i^p\right) \le g\left(\mathbb{E}\sum_{i=1}^n X_i^p\right),$$

which is precisely the desired inequality.

1.16 (Paley-Zygmund inequality) Markov inequality says a random variable is unlikely to be much bigger than its expectation. But what about the reverse? Can a nonnegative random variable be much smaller than its expectation with high probability? In general, yes (example?), but not if the second moment isn't too large. Let X be a nonnegative random variable with finite variance. Show that for any $\varepsilon \in [0,1]$:

$$\mathbb{P}\{X > \varepsilon \mathbb{E}X\} \geq (1-\varepsilon)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}.$$

Solution: Let E be the event $\{X > \varepsilon \mathbb{E}[X]\}$. Since the random variable $X - \varepsilon \mathbb{E}[X]$ is non-positive on the complement E^c, we have:

$$(1-\varepsilon)\mathbb{E}[X] \leq \mathbb{E}[(X - \varepsilon \mathbb{E}[X])\mathbf{1}_E].$$

By the Cauchy-Schwarz inequality,

$$(\mathbb{E}[(X - \varepsilon \mathbb{E}[X])\mathbf{1}_E])^2 \leq \mathbb{E}\left[(X - \varepsilon \mathbb{E}[X])^2\right] \mathbb{E}[\mathbf{1}_E^2] = \mathbb{E}\left[(X - \varepsilon \mathbb{E}[X])^2\right] \mathbb{P}(E).$$

Combining these inequalities leads to

$$(1-\varepsilon)^2 (\mathbb{E}[X])^2 \leq \mathbb{E}\left[(X - \varepsilon \mathbb{E}[X])^2\right] \mathbb{P}(E).$$

Finally, we use the fact that for $\varepsilon \in [0,1]$,

$$\mathbb{E}\left[(X - \varepsilon \mathbb{E}[X])^2\right] = \mathbb{E}[X^2] - (2\varepsilon - \varepsilon^2)(\mathbb{E}[X])^2 \leq \mathbb{E}[X^2].$$

Substituting this bound and rearranging the terms gives the desired result.

1.17 (Comparison of the ℓ^p norms) Let $0 \leq p \leq q \leq \infty$.

(a) Prove that for any vector $x \in \mathbb{R}^n$ we have

$$\|x\|_q \leq \|x\|_p \leq n^{\frac{1}{p} - \frac{1}{q}} \|x\|_q.$$

(b) Demonstrate that both inequalities in part (a) can be optimal. Specifically, find nonzero

vectors $x, y \in \mathbb{R}^n$ satisfying $\|x\|_p = \|x\|_q$ and $\|y\|_p = n^{\frac{1}{p}-\frac{1}{q}}\|y\|_q$.

Solution: (a) We only consider the case $0 < p < q \leq \infty$. The case $x = 0$ is trivial, so we assume $x \neq 0$. We first prove $\|x\|_p \geq \|x\|_q$. WLOG, assume $\|x\|_p = 1$, which implies $|x_i| \leq 1$ for all i. Since $p \leq q$, it follows that $|x_i|^q \leq |x_i|^p$. Summing over i gives $\|x\|_q^q \leq \|x\|_p^p = 1$, so $\|x\|_q \leq 1$. The general result follows from the homogeneity of norms. To prove $\|x\|_p \leq n^{1/p-1/q}\|x\|_q$, we simply let define the random variable X such that $\mathbb{P}(X = x_i) = 1/n$. Then we finish the proof using the fact $\|X\|_{L_p} \leq \|X\|_{L_q}$.

(b) Let e_i denote the i-th standard basis vector in $\mathbb{R}^n$ then apparently for any $i \in \{1, \dots n\}$, $\|e_i\|_p = \|e_i\|_q$. Let $\mathbf{1}_n$ denote the all-one vector in $\mathbb{R}^n$, then $\|\mathbf{1}_n\|_p = n^{1/p-1/q}\|\mathbf{1}_n\|_q$.

1.19 (Duality of the ℓ_p norms) Let $p, p' \in [1, \infty]$ be conjugate exponents.

(a) Show that Hölder inequality is tight: for any vector x there exists a vector $y \neq 0$ for which $\langle x, y\rangle = \|x\|_p\|y\|_{p'}$.

(b) Conclude that for every vector $x \in \mathbb{R}^n$, we have

$$\max\left\{\langle x, y\rangle : \ y \in B_{p'}^n\right\} = \|x\|_p.$$

Solution: (a) WLOG, we may assume that $x_i \geq 0$ for all i. For a general vector x, we can choose the sign of each component y_i to match the sign of the corresponding x_i. In the case where $p = 1, p' = \infty$, set $y = (1, \dots, 1)$. In the case where $p = \infty, p' = 1$, set $y = (0, \dots, 0, 1, 0, \dots, 0)$ where the value 1 is at the coordinate i_0 for which $|x_{i_0}| = \|x\|_\infty$. In the case where $p, p' \in (1, \infty)$, set $y_i := |x_i|^{p/p'}$ for all i.

(b) Trivial.

2 Sums of Independent Random Variables

2.1 (Products of i.i.d. random variables do not concentrate) Many random variables concentrate around the mean; in particular, the probability of exceeding the mean is usually constant. Here is an example of poor concentration, where this probability is exponentially small. Let $X_1, \dots, X_n$ be independent random variables uniformly distributed in $[0, 1]$. Prove that their

product $Y_n := X_1 \cdots X_n$ satisfies

$$(0.5)^n \leq \mathbb{P}\{Y_n \geq \mathbb{E}Y_n\} \leq (0.95)^n.$$

Solution: By independence, the expectation of Y_n is $\mathbb{E}[Y_n] = \prod_{i=1}^n \mathbb{E}[X_i] = \left(\frac{1}{2}\right)^n$. For the lower bound, we observe that the event $\{X_i \geq 1/2 \text{ for all } i\}$ implies that $Y_n = \prod_{i=1}^n X_i \geq (1/2)^n = \mathbb{E}[Y_n]$. This gives the bound:

$$\mathbb{P}(Y_n \geq \mathbb{E}[Y_n]) \geq \mathbb{P}(X_i \geq 1/2 \text{ for all } i = 1, \ldots, n) = \prod_{i=1}^n \mathbb{P}(X_i \geq 1/2) = \left(\frac{1}{2}\right)^n.$$

For the upper bound, we apply Markov's inequality to the non-negative random variable $\sqrt{Y_n}$:

$$\mathbb{P}(Y_n \geq \mathbb{E}[Y_n]) = \mathbb{P}\left(\sqrt{Y_n} \geq \sqrt{\mathbb{E}[Y_n]}\right) \leq \frac{\mathbb{E}[\sqrt{Y_n}]}{\sqrt{\mathbb{E}[Y_n]}}.$$

The numerator can be computed using independence:

$$\mathbb{E}[\sqrt{Y_n}] = \mathbb{E}\left[\prod_{i=1}^n \sqrt{X_i}\right] = \prod_{i=1}^n \mathbb{E}[\sqrt{X_i}] = \left(\frac{2}{3}\right)^n.$$

Substituting this and the value for $\sqrt{\mathbb{E}[Y_n]} = (1/\sqrt{2})^n$ into the inequality yields the result:

$$\mathbb{P}(Y_n \geq \mathbb{E}[Y_n]) \leq \frac{(2/3)^n}{(1/\sqrt{2})^n} = \left(\frac{2\sqrt{2}}{3}\right)^n \leq (0.95)^n.$$

2.2 (Gaussian tails: a lower bound) The lower bound in Proposition 2.1.2 was left unproved; let us prove it now. Let $g \sim N(0,1)$. Show that for all $t > 0$, we have

$$\mathbb{P}\{g \geq t\} \geq \frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

To do this, consider the difference between the left and right hand sides as a function of t. Check that this function decreases to 0 as t increases to infinity.

Solution: We consider the difference between the left and right-hand sides:

$$f(t) := \mathbb{P}(g \geq t) - \frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = 1 - \int_{-\infty}^t \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx - \frac{t}{t^2+1} \cdot \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Taking derivative,

$$f'(t) = -\frac{1}{\sqrt{2\pi}} e^{-t^2/2} - \frac{1 - 2t^2 - t^4}{(t^2+1)^2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} = -\frac{2}{(t^2+1)^2} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}.$$

Therefore, $f'(t) \le 0$ for all $t \ge 0$, and thus f decreases on $[0, \infty)$. We have $f(t) \to 0$ as $t \to \infty$. It follows that $f(t) \ge 0$ for any $t \ge 0$, and we finish the proof.

2.3 (Mills ratio) Although the Gaussian tail $\mathbb{P}\{g > t\}$ for $g \sim N(0,1)$ cannot be calculated analytically for all t, it can be expanded into series as follows:

$$\frac{\mathbb{P}\{g > t\}}{f(t)} = \frac{1}{t} - \frac{1}{t^3} + \frac{1 \cdot 3}{t^5} - \frac{1 \cdot 3 \cdot 5}{t^7} + \cdots \quad \text{for } t > 1, \tag{1}$$

where $f(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ is the density of g. Moreover, the ratio is sandwiched between any pair of consecutive partial sums of this series. For example,

$$\frac{1}{t} - \frac{1}{t^3} \le \frac{\mathbb{P}\{g > t\}}{f(t)} \le \frac{1}{t} - \frac{1}{t^3} + \frac{3}{t^5} \quad \text{for } t > 0. \tag{2}$$

Prove the sandwiching inequality by doing the following steps:

(a) Check that $f'(x) + xf(x) = 0$ for all x.

(b) Use the equation from part (a) to integrate $\mathbb{P}\{g > t\} = \int_t^\infty f(x)\,dx$ by parts. Repeat.

Solution: (a) Trivial.

(b) Using the identity $f(x) = -f'(x)/x$, we integrate by parts:

$$\mathbb{P}\{g > t\} = \int_t^\infty -\frac{1}{x} f'(x)\,dx = \left[-\frac{f(x)}{x}\right]_t^\infty - \int_t^\infty \frac{f(x)}{x^2}\,dx = \frac{f(t)}{t} - \int_t^\infty \frac{f(x)}{x^2}\,dx.$$

Repeat,

$$\int_t^\infty \frac{f(x)}{x^2}\,dx = \int_t^\infty -\frac{1}{x^3} f'(x)\,dx = \frac{f(t)}{t^3} - \int_t^\infty \frac{3f(x)}{x^4}\,dx.$$

Substituting this back gives:

$$\mathbb{P}\{g > t\} = \frac{f(t)}{t} - \frac{f(t)}{t^3} + \int_t^\infty \frac{3f(x)}{x^4}\,dx.$$

We can drop the $\int_t^\infty 3f(x)/x^4\,dx$ term to obtain:

$$\mathbb{P}\{g > t\} \geq \frac{f(t)}{t} - \frac{f(t)}{t^3}.$$

For the upper bound, we integrate the remainder term by parts one more time:

$$\int_t^\infty \frac{3f(x)}{x^4}\,dx = \int_t^\infty -\frac{3}{x^5} f'(x)\,dx = \frac{3f(t)}{t^5} - \int_t^\infty \frac{15f(x)}{x^6}\,dx.$$

Substituting this into the expression for $\mathbb{P}\{g > t\}$ yields:

$$\mathbb{P}\{g > t\} = \frac{f(t)}{t} - \frac{f(t)}{t^3} + \frac{3f(t)}{t^5} - \int_t^\infty \frac{15f(x)}{x^6}\,dx.$$

Dropping the non-positive remainder gives the inequality:

$$\mathbb{P}\{g > t\} \leq \frac{f(t)}{t} - \frac{f(t)}{t^3} + \frac{3f(t)}{t^5}.$$

2.7 (Small ball probability) Let $X_1, \ldots, X_N$ be nonnegative independent random variables with continuous distributions. Assume that the probability density functions of X_i are all uniformly bounded by K. Show that for any $\varepsilon > 0$ we have

$$\mathbb{P}\left\{\sum_{i=1}^N X_i \leq \varepsilon N\right\} \leq (eK\varepsilon)^N.$$

Solution: We prove the desired inequality with the exponential moment methods. Specifically,

$$\begin{aligned}
\mathbb{P}\left\{\sum_{i=1}^N X_i \leq \varepsilon N\right\} &= \mathbb{P}\left\{\sum_{i=1}^N -X_i \geq -\varepsilon N\right\} \\
&= \mathbb{P}\left\{\exp\left(\sum_{i=1}^N -X_i/\varepsilon\right) \geq -\exp(N)\right\} \\
&\leq \frac{\mathbb{E}\exp\left(\sum_{i=1}^N -X_i/\varepsilon\right)}{-\exp(N)} \\
&= \left[e\mathbb{E}\exp(-X_i/\varepsilon)\right]^N
\end{aligned}$$

Next, we bound the expectation term. Let the probability density function of X_1 be $f(x)$, which is bounded by a constant K. The random variable $Y = X_1/\varepsilon$ has a density $g(y) = \varepsilon f(\varepsilon y)$, which

is thus bounded by εK. Since X_1 is non-negative, we have

$$\mathbb{E}\left[e^{-X_1/\varepsilon}\right] = \int_0^\infty e^{-y} g(y) dy \leq (\varepsilon K) \int_0^\infty e^{-y} dy = \varepsilon K.$$

Substituting this into our main inequality yields the desired result.

2.8 (An MGF comparison inequality) Let X and Y be random variables with the same mean. Assume that X takes values in an interval $[a, b]$, while Y takes values in the two-point set $\{a, b\}$. Prove that

$$\mathbb{E}\, e^{\lambda X} \leq \mathbb{E}\, e^{\lambda Y} \quad \text{for all } \lambda \in \mathbb{R}.$$

Solution: WLOG, we can assume $a = 0$, $b = 1$. Since X and Y have the same mean, $\mathbb{P}(Y = 1) = \mathbb{E}X$ and $\mathbb{P}(Y = 0) = 1 - \mathbb{E}X$. For $\lambda \geq 0$, $f(x) = e^{\lambda x}$ is convex, we observe

$$e^{\lambda X} \leq X + (1 - X)e.$$

Taking expectations, we have

$$\mathbb{E}e^{\lambda X} \leq \mathbb{E}X + (1 - \mathbb{E}X)e = \mathbb{E}e^{\lambda Y}.$$

We have proved the desired inequality.

2.12 (Reverse Chernoff) Show that a Binomial random variable $S_N \sim \text{Binom}(N, \mu/N)$ satisfies the "reverse Chernoff" inequality:

$$\mathbb{P}\{S_N = t\} \geq e^{-\mu} \left(\frac{\mu}{t}\right)^t \quad \text{for every integer } t \in [\mu, N].$$

Solution: As $S_N \sim \text{Binom}(N, \mu/N)$,

$$\begin{aligned}
\mathbb{P}(S_N = t) &= \binom{N}{t} \left(\frac{\mu}{N}\right)^t \left(\frac{N-\mu}{N}\right)^{N-t} \\
&\geq \left(\frac{N}{t}\right)^t \left(\frac{\mu}{N}\right)^t \left(\frac{N-\mu}{N}\right)^{N-t} \\
&= \left(\frac{\mu N}{t(N-\mu)}\right)^t \left(1 - \frac{\mu}{N}\right)^N \\
&\geq \left(\frac{\mu}{t}\right)^t e^{-\mu}.
\end{aligned}$$

Here the first inequality holds using Exercise 0.6, and the last inequality holds by $N/(N-\mu) \geq 1$ and $(1-\mu/N)^N \leq e^{-\mu}$.

2.13 (Poisson tails) Thanks to the Poisson limit theorem (Theorem 1.7.6), it is reasonable to expect that Chernoff inequality works for Poisson variables too. And it does! Use the exponential moment method to prove the following tail bounds for $X \sim \text{Pois}(\mu)$:

(a) (Version of Theorem 2.3.1) $\mathbb{P}\{X \geq t\} \leq e^{-\mu}(e\mu/t)^t$ for every $t \geq \mu$.

(b) (Version of Remark 2.3.2) $\mathbb{P}\{X \leq t\} \leq e^{-\mu}(e\mu/t)^t$ for every $0 < t \leq \mu$.

(c) (Version of Corollary 2.3.4) $\mathbb{P}\{|X-\mu| \geq \delta\mu\} \leq 2\exp(-\delta^2\mu/3)$ for every $0 \leq \delta \leq 1$.

(d) (Version of Exercise 2.12) $\mathbb{P}\{X = t\} \geq e^{-\mu}(\mu/t)^t$ for every integer $t > 0$.

Solution: (a) We apply the standard exponential moment method. For any $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}(X \geq t) &= \mathbb{P}(\exp(\lambda X) \geq \exp(\lambda t)) \\ &\leq \frac{\mathbb{E}(\exp(\lambda X))}{\exp(\lambda t)} \\ &= \frac{\exp(\mu(e^\lambda - 1))}{\exp(\lambda t)}.\end{aligned}$$

Setting $\lambda = \log(t/\mu)$ yields

$$\mathbb{P}(X \geq t) \leq \frac{\exp(\mu(e^\lambda - 1))}{\exp(\lambda t)} = e^{-\mu}\left(\frac{e\mu}{t}\right)^t,$$

which is exactly the desired inequality.

(b) We consider the event $\{-X \geq -t\}$ and then apply the standard exponential moment method. For any $\lambda > 0$, we have

$$\begin{aligned}\mathbb{P}(X \geq t) &= \mathbb{P}(-X \geq -t) \\ &= \mathbb{P}(\exp(-\lambda X) \geq \exp(-\lambda t)) \\ &\leq \frac{\mathbb{E}(\exp(-\lambda X))}{\exp(-\lambda t)} \\ &= \frac{\exp(\mu(e^{-\lambda} - 1))}{\exp(-\lambda t)}.\end{aligned}$$

Setting $\lambda = -\log(t/\mu)$ yields

$$\mathbb{P}(X \geq t) \leq \frac{\exp(\mu(e^{-\lambda} - 1))}{\exp(-\lambda t)} = e^{-\mu} \left(\frac{e\mu}{t}\right)^t,$$

which is exactly the desired inequality.

(c) Using results in part (a), we have

$$\mathbb{P}(X \geq (1+\delta)\mu) \leq \frac{e^{\delta\mu}}{(1+\delta)^{(1+\delta)\mu}} = \exp\left(-\mu((1+\delta)\log(1+\delta) - \delta)\right).$$

Note that $\delta \leq 1$, we expand the expression inside exponent into a Taylor series:

$$(1+\delta)\log(1+\delta) - \delta = \frac{\delta^2}{2} - \frac{\delta^3}{2\cdot 3} + \frac{\delta^4}{3\cdot 4} - \frac{\delta^5}{4\cdot 5} + \cdots \geq \frac{\delta^2}{3}.$$

Plug this into the previous bounds and get

$$\mathbb{P}(X \geq (1+\delta)\mu) \leq \exp\left(\frac{-\mu\delta^2}{3}\right)$$

The left tail can be bounded similarly with results in part (b). We complete the proof by combining the two tails with a union bound.

(d) Trivial.

2.14 (Chernoff inequality: small deviations) Let's extend Corollary 2.3.4. Let X_i be independent Bernoulli random variables with parameters p_i. Consider their sum $S_N = \sum_{i=1}^N X_i$ and denote its mean by $\mu = \mathbb{E}S_N$. Show that

$$\mathbb{P}\{|S_N - \mu| \geq \delta\mu\} \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right) \quad \text{for every } \delta \geq 0.$$

Solution: Using Theorem 2.3.1 with $t = (1+\delta)\mu$ and simplifying, we get

$$\mathbb{P}\{S_N \geq (1+\delta)\mu\} \leq \exp\left[-\mu\left((1+\delta)\log(1+\delta) - \delta\right)\right].$$

To simplify the exponent, we use the numerical inequality $\log(1+x) \geq x/(1+x/2)$ for any $x \geq 0$. To prove this, we Let $g(x) = (x+2)\log(x+1) - x$. We need to show $g(x) \geq 0$ for any $x \geq 0$. Indeed, we have $g'(x) = 1/(x+1) + \log(x+1)$, which is non-negative for any $x \geq 0$. Since $g(x) = 0$, we

finish the proof. Applying this inequality

$$\begin{aligned}\mathbb{P}\{S_N \geq (1+\delta)\mu\} &\leq \exp\left[-\mu\left((1+\delta)\delta/(1+\delta/2)-\delta\right)\right] \\ &\leq \exp\left[-\mu\left(\delta^2/(2+\delta)\right)\right].\end{aligned}$$

The lower tail can be bounded in a similar manner. Then we complete the proof by combining the two tail bounds with a union bound argument.

2.17 (No mean estimator is subgaussian everywhere) You may wonder if Theorem 2.4.1 holds for all $t > 0$. It does not. In fact, no mean estimator can provide subgaussian confidence for all quantities. *Disprove* the following claim:

> "There exists an absolute constant $c > 0$ such that the following holds. For any integer $N > 0$ and every $t \geq 0$, one can find a function $\widehat{\mu} : \mathbb{R}^N \to \mathbb{R}$ satisfying
>
> $$\mathbb{P}\left\{|\widehat{\mu}(X_1, \ldots, X_N) - \mu| \geq \frac{t\sigma}{\sqrt{N}}\right\} \leq 2e^{-ct^2}$$
>
> for all i.i.d. random variables $X_1, \ldots, X_N$ with mean μ and variance σ^2."

Do this by working out a version of *Le Cam's two-point method*:

(a) Consider two random vectors $X = (X_1, \ldots, X_N)$ and $Y = (Y_1, \ldots, Y_N)$ with independent Laplace[1] Check that $\mathbb{P}\{X \in B\} \leq e^{\mu}\mathbb{P}\{Y \in B\}$ for any measurable subset $B \subset \mathbb{R}^N$.

(b) Assume for contradiction that the claim holds. Apply it to bound the probabilities of the events $|\widehat{\mu}(X)| \geq \mu/2$ and $|\widehat{\mu}(Y) - \mu| \geq \mu/2$.

(c) Choose μ large enough and replace Y with X using (a) to make both events $|\widehat{\mu}(X)| < \mu/2$ and $|\widehat{\mu}(X) - \mu| < \mu/2$ likely. Use triangle inequality to arrive at a contradiction.

Solution: (a) The density of X_1 and Y_1 are $f(x_1) = \exp(-|x_1|)/2$, $g(y_1) = \exp(-|y_1 - \mu|)/2$. By the triangular inequality, we have $f(x_1) \leq e^{\mu} g(x_1)$, $\forall x_1 \in \mathbb{R}$. Because both $X_1, ..., X_N$ and $Y_1, ..., Y_N$ are i.i.d., the joint density are $f_X(x) = \prod_{i=1}^N f(x_i)$, $g_Y(y) = \prod_{i=1}^N g(y_j)$, respectively. Thus, $f_X(x) \leq e^{N\mu} g_Y(x)$, $\forall x \in \mathbb{R}^N$. For any measurable $B \subset \mathbb{R}^N$,

$$\mathbb{P}(X \in B) = \int_B f(x)dx \leq \int_B e^{N\mu} g(x)dx = e^{N\mu}\mathbb{P}(Y \in B).$$

[1] coordinates: $X_i \sim \text{Lap}(0, 1)$ and $Y_i \sim \text{Lap}(\mu, 1)$ for some $\mu > 0$.

(b) Assuming the claim is true, we get

$$\mathbb{P}(|\hat{\mu}(X)| \geq \mu/2) \leq 2e^{-cN\mu^2/16},$$

and similarly,

$$\mathbb{P}(|\hat{\mu}(Y) - \mu| \geq \mu/2) \leq 2e^{-cN\mu^2/16}.$$

(c) By part (a),

$$\mathbb{P}(|\hat{\mu}(X) - \mu| \geq \mu/2) \leq e^{N\mu}\mathbb{P}(|\hat{\mu}(Y) - \mu| \geq \mu/2) \leq 2e^{-cN(\mu^2 - 16\mu/c)/16}.$$

Setting $\mu = K/c$, we obtain

$$\mathbb{P}(|\hat{\mu}(X) - \mu| \geq \mu/2) \leq 2e^{-N(K^2-16K)/16c}.$$

However,

$$\mathbb{P}(|\hat{\mu}(X)| < \mu/2) \geq 1 - 2e^{-NK^2/16c}.$$

One must have $1 - 2e^{-NK^2/16c} \leq 2e^{-N(K^2-16K)/16c}$. Since K can be chosen arbitrarily large, the left-hand side tends to 1, while the right-hand side tends to 0. This is impossible.

2.18 (Most degrees of sparse random graphs are OK) In Exercise 1.10, we saw that a typical sparse random graph $G \sim G(n,p)$ has isolated vertices ("friendless students"), resulting in a minimum degree of zero. However, most vertex degrees are still close to the expected degree $d = (n-1)p$. Prove that there exists an absolute constant $C > 0$ such that if $d \geq C$, then with probability at least 0.99 the following event holds: at least 99% of the vertices have their degrees between $0.9d$ and $1.1d$.

Solution: Call a vertex *bad* if its degree lies outside $[0.9d, 1.1d]$. Let X denote the number of bad vertices. Then

$$\begin{aligned}
\mathbb{E}X &= \mathbb{E}\sum_{i=1}^{n} \mathbb{1}\{\text{vertex } i \text{ is bad}\} \\
&= \sum_{i=1}^{n} \mathbb{E}\mathbb{1}\{\text{vertex } i \text{ is bad}\} \\
&= \sum_{i=1}^{n} \mathbb{P}(|d_i - d| > 0.1d)
\end{aligned}$$

$$\leq 2ne^{-cd},$$

where d_i denotes the degree of vertex i, and the final bound follows from a Chernoff inequality (Corollary 2.3.4). By Markov's inequality,

$$\mathbb{P}(X \geq 0.01n) \leq \frac{\mathbb{E}X}{0.01n} \leq \frac{2ne^{-cd}}{0.01n} = 200e^{-cd}.$$

Thus, if $d \geq \log 20000/c$, then $\mathbb{P}(X \geq 0.01n) \leq 0.01$.

2.19 (Maximum degrees of sparse random graphs) In Exercise 1.10, we observed that a typical sparse random graph $G \sim G(n,p)$ has the minimum degree zero. But what about the maximum degree ("the number of friends of the most popular student"), denoted $\Delta(G)$? Show that there exist absolute constants $c, c_1, c_2 > 0$ such that the following holds: If $n \geq 3$ and the expected degree $d = (n-1)p$ satisfies $d \leq c(\log n)^{0.99}$, then

$$c_1 \frac{\log n}{\log\log n} \leq \Delta(G) \leq c_2 \frac{\log n}{\log\log n}$$

with probability at least 0.99.

Solution:

Upper Bound We first show that if K is a sufficiently large constant, $\Delta(G) \leq K \log n/\log\log n$ with probability at least 0.995.

$$\begin{aligned}
\mathbb{P}\left(\Delta(G) > \frac{K\log n}{\log\log n}\right) &= \mathbb{P}\left(\exists \text{ vertex } i, d_i > \frac{K\log n}{\log\log n}\right) \\
&\leq \sum_{i=1}^{n} \mathbb{P}\left(d_i > \frac{K\log n}{\log\log n}\right) \\
&\leq ne^{-d}\left(\frac{ed}{K\log n/\log\log n}\right)^{\frac{K\log n}{\log\log n}} \\
&= \exp\left(\log n - d + \frac{K\log n}{\log\log n} + \frac{K\log n\log d}{\log\log n} - \frac{K\log n}{\log\log n}\log\left(\frac{K\log n}{\log\log n}\right)\right)
\end{aligned}$$

The first inequality uses a union bound argument and the second inequality uses the Chernoff inequality. Next we analyze the exponent. We want to show that if we set K as a large enough

constant the exponent is always below -6. Let $x = \log n > 1$, the exponent can be rewritten as

$$g(x) = x - d + \frac{Kx}{\log x} + \frac{Kx \log d}{\log x} - \frac{Kx}{\log x}(\log K + \log x - \log\log x)$$

Since $d \le cx^{0.99}$,

$$\begin{aligned} g(x) &\le x + \frac{Kx}{\log x} + \frac{Kx(0.99\log x + \log c)}{\log x} - \frac{Kx}{\log x}(\log K + \log x - \log\log x) \\ &= x + \frac{Kx}{\log x}(-0.01\log x + \log c - \log K + \log\log x) \\ &= (1 - 0.01K)x + \frac{Kx}{\log x}(\log c - \log K + \log\log x). \end{aligned}$$

A direct but tedious check shows that if $K > e^{100} + 1000c + c^{10}$, then this upper bound of $g(x)$ is below -6 for all $x > 1$.

Lower Bound To avoid tricky cases such as empty graphs, we show that for some constant K', then $\Delta(G) \ge K' \log n / \log\log n$ with probability at least 0.995 when n is sufficiently large. WLOG we assume n is even. Divide the set of vertices into two subsets V_1 and V_2 of the same size. For each vertex $i \in V_2$, we use $\tilde{d}_i$ to denote the number of vertices in V_1 connected to i. For vertex $i \in V_2$, $\tilde{d}_i$s are i.i.d. and $\tilde{d}_i \le d_i$. Define $\tilde{d} = \mathbb{E}d_i = d/2$,

$$\begin{aligned} \mathbb{P}\left(\Delta(G) < \frac{K'\log n}{\log\log n}\right) &= \mathbb{P}\left(\text{for all vertex } i, d_i < \frac{K'\log n}{\log\log n}\right) \\ &\le \mathbb{P}\left(\text{for all vertex } i \in V_2, \tilde{d}_i < \frac{K'\log n}{\log\log n}\right) \\ &= \prod_{i\in V_2} \mathbb{P}\left(\tilde{d}_i < \frac{K'\log n}{\log\log n}\right) \\ &= \left[\mathbb{P}\left(\tilde{d}_1 < \frac{K'\log n}{\log\log n}\right)\right]^{n/2} \\ &\le \left[1 - \mathbb{P}\left(\tilde{d}_1 = \frac{K'\log n}{\log\log n}\right)\right]^{n/2} \\ &\le \exp\left(-\mathbb{P}\left(\tilde{d}_1 = \frac{K'\log n}{\log\log n}\right)\cdot(n/2)\right) \end{aligned}$$

We analyze the logarithm of the term in the exponent: $\mathbb{P}\left(\tilde{d}_1 = K'\log n/\log\log n\right)\cdot(n/2)$. Using

the Poisson approximation, for sufficiently large n, there exists a constant C_0 such that

$$\mathbb{P}\left(\tilde{d}_i = \frac{K' \log n}{\log\log n}\right) \geq C_0 e^{-\tilde{d}} \left(\frac{\tilde{d}}{K' \log n/\log\log n}\right)^{\frac{K' \log n}{\log\log n}}.$$

Then we have:

$$\begin{aligned}
&\log\left[\mathbb{P}\left(\tilde{d}_1 = K' \log n/\log\log n\right) \cdot (n/2)\right] \\
&\geq \left(\log C_0 - \tilde{d} + \frac{K' \log n \log \tilde{d}}{\log\log n} - \frac{K' \log n}{\log\log n} \log\left(\frac{K' \log n}{\log\log n}\right) + \log n - \log 2\right).
\end{aligned}$$

Let $x = \log n > 1$, the lower bound can be rewritten as

$$h(x) = \log C_0 - \tilde{d} + \frac{K' x \log \tilde{d}}{\log x} - \frac{K' x}{\log x}\left(\log K' + \log x - \log\log x\right) + x - \log 2.$$

Recall that $\tilde{d} = d/2 \leq c x^{0.99}/2$, the dominant term in $h(x)$ is $(1-K')x$. If we choose a small positive constant K' (e.g., $K' < 0.01$), $h(x)$ tends to ∞ as $n \to \infty$. Consequently $\mathbb{P}\left(\Delta(G) < K' \log n/\log\log n\right) \to 0$, and for sufficiently large n, $\mathbb{P}\left(\Delta(G) < K' \log n/\log\log n\right) \leq 0.005$.

2.20 (Expansion property of random graphs) Random graphs $G(n,p)$ exhibit a remarkable property: the number of edges $e(S,T)$ connecting any two vertex subsets S and T is proportional to the sizes of these sets, provided they are not too small. Prove that there exists an absolute constant $C > 0$ such that, with probability at least $1 - 2^{-n}$, the following event occurs:

$$0.9p \leq \frac{e(S,T)}{|S||T|} \leq 1.1p \quad \text{for all disjoint subsets of vertices } S, T \text{ with } |S||T| \geq \frac{Cn}{p}.$$

Solution: We observe that $\mathbb{E}e(S,T) = p|S||T|$. For any disjoint subsets S, T with $|S||T| \geq 1000n/p$, Corollary 2.3.4 gives

$$\mathbb{P}\big(|e(S,T) - p|S||T|| \geq 0.1p|S||T|\big) \leq 2e^{-0.01p|S||T|/3} \leq 2e^{-10n}.$$

There are at most 2^n pairs of such subsets. Hence, by a union bound,

$$\begin{aligned}
&\mathbb{P}\big(\exists S, T : |S||T| \geq 1000n/p, |e(S,T) - p|S||T|| \geq 0.1p|S||T|\big) \\
&\leq \sum_{S,T} \mathbb{P}(|e(S,T) - p|S||T|| \geq 0.1p|S||T|)
\end{aligned}$$

$$\leq 2^n \cdot 2e^{-10n}$$
$$\leq 2^{-n}.$$

This establishes the claim.

2.23 (Subgaussian MGF requires zero mean) You might wonder why we assumed that $\mathbb{E}X = 0$ in property (iv) of Proposition 2.6.1. Show that any random variable X satisfying this property *must* have zero mean.

Solution: Let us use the numeric inequality:

$$1 + x \leq e^x \leq 1 + x + \frac{x^2}{2} e^{|x|}.$$

Suppose X satisfies property (iv), we have

$$1 + \lambda \mathbb{E}X \leq \mathbb{E}\exp(\lambda X) \leq \mathbb{E}\exp(K^2\lambda^2) \leq 1 + K^2\lambda^2 + \frac{K^4\lambda^4}{2}\exp(K^2\lambda^2).$$

If $\mathbb{E}X > 0$, then for $0 < \lambda \leq \min\{1/100K, \mathbb{E}X/100K^2\}$, we have

$$1 + K^2\lambda^2 + \frac{K^4\lambda^4}{2}\exp(K^2\lambda^2) \leq 1 + 2K^2\lambda^2 \leq 1 + \lambda\mathbb{E}X,$$

which leads to a contradiction. Otherwise, if $\mathbb{E}X < 0$, then for $0 > \lambda \geq \max\{-1/100K, \mathbb{E}X/100K^2\}$, we have

$$1 + K^2\lambda^2 + \frac{K^4\lambda^4}{2}\exp(K^2\lambda^2) \leq 1 + 2K^2\lambda^2 \leq 1 + \lambda\mathbb{E}X,$$

which also leads to a contradiction. Therefore, we finish the proof.

2.26 (Subgaussian characterization: MGF of X^2) In Proposition 2.6.1 we saw a few equivalent ways to describe subgaussian distributions. Here is one more. Prove that a random variable X is subgaussian if and only if there exists $K > 0$ such that

$$\mathbb{E}\exp(\lambda^2 X^2) \leq \exp(\lambda^2 K^2) \quad \text{for all } \lambda \text{ such that } |\lambda| \leq \frac{1}{K}. \tag{2.31}$$

More precisely, show that if X is subgaussian, then (2.31) holds with $K = \|X\|_{\psi_2}$. Conversely, prove that if (2.31) holds, then X is subgaussian with $\|X\|_{\psi_2} \leq 2K$.

Solution: For simplicity of notation, we use C_0 to denote $\|X\|_{\psi_2}$. Suppose X is subgaussian, then when $|\lambda| \le 1/C_0$, $C_0^2\lambda^2 \le 1$, the function $f(x) = x^{C_0^2\lambda^2}$ is concave. By Jensen's inequality, for any $|\lambda| \le 1/C_0$,

$$\mathbb{E}\exp(\lambda^2 X^2) = \mathbb{E}[\exp(X^2/C_0^2)]^{C_0^2\lambda^2} \le [\mathbb{E}\exp(X^2/C_0^2)]^{C_0^2\lambda^2} \le 2^{C_0^2\lambda^2} \le \exp(\lambda^2 C_0^2).$$

Therefore, (2.31) holds with $K = C_0$. On the other hand, suppose (2.31) holds, substituting $\lambda = 1/2K$, we have

$$\mathbb{E}\exp(X^2/(2K)^2) \le \exp(K^2/(2K)^2) = \exp(1/4) \le 2.$$

Hence X is subgaussian and $\|X\|_{\psi_2} \le 2K$.

2.27 (Subgaussian characterization: almost stochastic dominance) For two random variables X and Y, let us say that $X \preceq Y$ if

$$\mathbb{P}\{X \ge t\} \le 2\mathbb{P}\{Y \ge t\} \quad \text{for all } t \in \mathbb{R}. \tag{2.27.1}$$

(a) Prove that X is subgaussian if and only if there exists $K > 0$ such that

$$|X| \preceq K|g| \quad \text{where } g \sim N(0,1). \tag{2.27.2}$$

More precisely, show that if X is subgaussian, then (2.27.2) holds with some $K \le C\|X\|_{\psi_2}$. Conversely, prove that if (2.27.2) holds, then X is subgaussian with $\|X\|_{\psi_2} \le CK$. As always, C stands for an absolute constant of your choice.

(b) Show by example that part (a) may fail if the factor 2 in (2.27.1) is replaced with 1.

Solution: (a) According to Proposition 2.6.6, every subgaussian random variable X satisfies

$$\mathbb{P}\{|X| \ge t\} \le 2\exp\left(-ct^2/\|X\|_{\psi_2}^2\right) \text{ for all } t \ge 0.$$

On the other hand, by Proposition 2.1.2,

$$\mathbb{P}\{K|g| \ge t\} \ge \frac{1}{\sqrt{2\pi}}\frac{2Kt}{t^2+K^2}\exp\left(-t^2/2K^2\right)$$

$$\begin{aligned}
&\geq \frac{1}{\sqrt{2\pi}} \frac{2t}{K} \exp\left(-t^2/2K^2 - t^2/K^2\right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{2t}{K} \exp\left(-3t^2/2K^2\right) \text{ for all } t \geq 0.
\end{aligned}$$

In the second inequality we use the numerical inequality $1 + x \leq e^x$ for all $x \in \mathbb{R}$. We split the interval $(0, \infty)$ into two parts: $(0, Kq_{0.5}]$ and $(Kq_{0.5}, \infty)$, where $q_{0.5}$ is the median of the random variable $|g|$. Then, for any $t \in (0, Kq_{0.5}]$, we always have

$$\mathbb{P}(|X| \geq t) \leq 1 \leq 2\mathbb{P}(K|g| \geq t).$$

For $t \in (Kq_{0.5}, \infty)$,

$$\begin{aligned}
\mathbb{P}\{K|g| \geq t\} &\geq \frac{1}{\sqrt{2\pi}} \frac{2t}{K} \exp\left(-3t^2/2K^2\right) \\
&\geq \frac{2q_{0.5}}{\sqrt{2\pi}} \exp\left(-3t^2/2K^2\right) \\
&\geq \exp\left(-7t^2/2K^2\right).
\end{aligned}$$

The final inequality holds since $q_{0.5} \approx 0.6745$,

$$\frac{2q_{0.5}}{\sqrt{2\pi}} \geq \exp(-2q_{0.5}^2) \geq \exp(-2t^2/K^2), \text{ for all } t \in (Kq_{0.5}, \infty).$$

Therefore, when X is subgaussian, then (2.27.2) holds as long as $K^2 \geq 7 \|X\|_{\psi_2}^2 / 2c$.

On the other hand, suppose (2.27.2) holds with constant K, we get

$$\mathbb{P}(|X| \geq t) \leq \mathbb{P}(K|g| \leq t) \leq 2 \exp\left(\frac{-3ct^2}{8K^2}\right).$$

The upper bound on $\mathbb{P}(K|g| \leq t)$ is from Proposition 2.6.6 and the fact that $\|g\|_{\psi_2} = \sqrt{8/3}$. Therefore, there exists constant C such that $\|X\|_{\psi_2} = CK$ by Proposition 2.6.6.

(b) We simply consider the X as a constant random variable ($X \equiv C_0 > 0$), which is subgaussian. However, for any constant K, we have

$$\mathbb{P}(|X| \geq t) = 1 > \mathbb{P}(K|g| \geq t) \text{ for all } t \leq C_0.$$

2.28 (Subgaussian characterization: convex dominance) For two random variables X and Y, let us say that $X \precsim Y$ if $\mathbb{E}\Phi(X) \leq \mathbb{E}\Phi(Y)$ for any convex, increasing function $\Phi : \mathbb{R} \to \mathbb{R}$. Prove

that X is subgaussian if and only if there exists $K > 0$ such that

$$|X| \precsim K|g| \quad \text{where } g \sim N(0,1). \tag{2.34}$$

More precisely, show that if X is subgaussian, then (2.34) holds with some $K \leq C\|X\|_{\psi_2}$. Conversely, prove that if (2.34) holds, then X is subgaussian with $\|X\|_{\psi_2} \leq CK$.

Solution: Without loss of generality, we can assume $\Phi(\cdot)$ is a convex, increasing function with $\Phi(0) = 0$ (otherwise, we simply replace $\Phi(x)$ with $\Phi(x) - \Phi(0)$). This ensures $\Phi(x) \geq 0$ for all $x \geq 0$. Furthermore, unless Φ is identically zero, it must be strictly increasing on $(0, \infty)$, which guarantees that its inverse, $\Phi^{-1}(\cdot)$, is well-defined. By the integral representation of expectation,

$$\mathbb{E}(\Phi(|X|)) = \int_0^\infty \mathbb{P}(\Phi(|X|) \geq t)dt = \int_0^\infty \mathbb{P}(|X| \geq \Phi^{(-1)}(t))dt,$$

Suppose X is subgaussian, using the results in Exercise 2.27, there exists a constant C_0 such that

$$\mathbb{P}(X \geq t) \leq 2\mathbb{P}(Y \geq t), \text{ for all } t \geq 0.$$

Applying this tail bound to the integral gives

$$\begin{aligned}
\mathbb{E}(\Phi(|X|)) \leq & \int_0^\infty 2\mathbb{P}(C_0 \|X\|_{\psi_2} |g| \geq \Phi^{(-1)}(t))dt \\
= & \int_0^\infty 2\mathbb{P}(\Phi(C_0 \|X\|_{\psi_2} |g|) \geq t)dt \\
= & 2\mathbb{E}(\Phi(C_0 \|X\|_{\psi_2} |g|))
\end{aligned}$$

Since $\Phi(\cdot)$ is convex,

$$2\mathbb{E}(\Phi(C_0 \|X\|_{\psi_2} |g|)) \leq \mathbb{E}(\Phi(2C_0 \|X\|_{\psi_2} |g|)).$$

It follows that $|X| \precsim 2C_0 \|X\|_{\psi_2} |g|$.

On the other hand, suppose $|X| \precsim K|g|$, then

$$\mathbb{E} \exp(3X^2/8K^2) \leq \mathbb{E} \exp(3g^2/8) = 2.$$

We conclude that $\|X\|_{\psi_2} \leq \sqrt{8/3}K$.

2.30 (A reverse Hoeffding inequality) Let $X_1, \dots, X_N$ be independent subgaussian random variables with zero means and **unit variances**, and let $a_1, \dots, a_N \in \mathbb{R}$. Prove that

$$\mathbb{P}\left\{\left|\sum_{i=1}^N a_i X_i\right| \geq \frac{1}{2}\left(\sum_{i=1}^N a_i^2\right)^{1/2}\right\} \geq \frac{c}{K^4}$$

where $K = \max_{i \leq N} \|X_i\|_{\psi_2}$ and $c > 0$ is an absolute constant. This result is a general, quantitative version of Example 1.5.1, though it does not attain the exact probability of $1/2$.

Solution: Define $Y \coloneqq |\sum_{i=1}^N a_i X_i|^2$, then Y is a non-negative random variable with $\mathbb{E}Y = \sum_{i=1}^N a_i^2$. By the Paley-Zygmund inequality,

$$\begin{aligned}\mathbb{P}\left\{\left|\sum_{i=1}^N a_i X_i\right| \geq \frac{1}{2}\left(\sum_{i=1}^N a_i^2\right)^{1/2}\right\} =& \mathbb{P}\left\{Y \geq \frac{1}{4}\left(\sum_{i=1}^N a_i^2\right)\right\} \\ \geq& \left(\frac{3}{4}\right)^2 \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2} \\ =& \frac{9\left(\sum_{i=1}^N a_i^2\right)^2}{16\mathbb{E}\left|\sum_{i=1}^N a_i X_i\right|^4}\end{aligned}$$

By Khintchine's inequality (Theorem 2.7.5), there exists an absolute constant $C > 0$ such that

$$\mathbb{E}\left|\sum_{i=1}^N a_i X_i\right|^4 \leq CK^4\left(\sum_{i=1}^N a_i^2\right)^2.$$

Substituting this upper bound into our previous result yields the desired inequality:

$$\mathbb{P}\left\{\left|\sum_{i=1}^N a_i X_i\right| \geq \frac{1}{2}\left(\sum_{i=1}^N a_i^2\right)^{1/2}\right\} \leq \frac{9}{16CK^4}.$$

2.31 (Hoeffding requires zero mean) Let $X_1, X_2, \dots$ be i.i.d. random variables that satisfy the Hoeffding-like inequality **(2.14)** for any N, any coefficient vector $a \in \mathbb{R}^N$, and for some $c > 0$ independent of N and a. Prove that $\mathbb{E}X_i = 0$.

Solution: Suppose $X_1, X_2, ..., X_N$ satisfies the inequality (2.14). Let $a_i = 1/N$, $i = 1, ..., N$, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^N \frac{1}{N} X_i\right| \geq t\right) \leq 2\exp\left(\frac{-ct^2}{1/N}\right).$$

According to Proposition 2.6.6,

$$\left\| \sum_{i=1}^{N} X_i/N \right\|_{\psi_2} \leq 1/N.$$

By centering (Lemma 2.7.8),

$$\left\| \sum_{i=1}^{N} X_i/N - \mathbb{E}X \right\|_{\psi_2} \leq C/N.$$

Therefore, we have

$$\|\mathbb{E}X_1\| \leq \left\| \sum_{i=1}^{N} X_i/N \right\|_{\psi_2} + \left\| \sum_{i=1}^{N} X_i/N - \mathbb{E}X \right\|_{\psi_2}.$$

Since N can be arbitrarily large, we must conclude $\mathbb{E}X_1 = 0$.

2.32 (Subgaussian norm of a sum) Prove that any two independent, mean zero, subgaussian random variables X and Y satisfy

$$\|X + Y\|_{\psi_2} \asymp \|X\|_{\psi_2} + \|Y\|_{\psi_2}$$

where $\asymp$ indicates equivalence up to absolute constant factors.

Solution: By triangular inequality, $\|X + Y\|_{\psi_2} \leq \|X\|_{\psi_2} + \|Y\|_{\psi_2}$. To establish a reverse inequality, it suffices by symmetry to show there exists a constant C such that for any two independent, mean zero, subgaussian r.v. X and Y, $\|X + Y\|_{\psi_2} \geq C \|X\|_{\psi_2}$. The MGF of $X + Y$ reads

$$\begin{aligned} \mathbb{E}\exp(\lambda(X+Y)) &= \mathbb{E}\exp(\lambda X) \cdot \mathbb{E}\exp(\lambda Y) \\ &\geq \mathbb{E}\exp(\lambda X) \cdot \exp(\lambda \mathbb{E}Y) \\ &= \mathbb{E}\exp(\lambda X). \end{aligned}$$

Since $\mathbb{E}\exp((X+Y)/\|X+Y\|_{\psi_2}) = 2$, we can obtain $\mathbb{E}\exp((X)/\|X+Y\|_{\psi_2}) \leq 2$, and thus $\|X + Y\|_{\psi_2} \geq \|X\|_{\psi_2}$.

2.33 (Subgaussian norm of i.i.d. sums) In light of the Pythagorean identity (2.19), it is natural to ask if the inequality in Proposition 2.7.1 can be reversed. In this and the next exercise, you will demonstrate that the answer is "yes" for identical distributions of X_i, but "no" in general.

(a) Let $X, X_1, X_2, \ldots, X_N$ be i.i.d. mean zero, subgaussian random variables. Prove that

$$\left\| \sum_{i=1}^{N} X_i \right\|_{\psi_2} \asymp \sqrt{N} \, \|X\|_{\psi_2}$$

where the sign "$\asymp$" indicates equivalence up to absolute constant factors.

(b) Deduce that the binomial random variable $S_N \sim \text{Binom}(N, p)$ satisfies

$$\|S_N - Np\|_{\psi_2} \asymp \sqrt{\frac{N}{\log(2/p)}}.$$

Solution: (a) According to Proposition 2.6.6,

$$\mathbb{E} \exp\left(\lambda \sum_{i=1}^{N} X_i\right) = \left[\mathbb{E} \exp\left(\lambda X\right)\right]^N \leq \exp(C\lambda^2 N \, \|X\|_{\psi_2}^2), \forall \lambda \in \mathbb{R}.$$

By proposition 2.6.6, up to absolute constant factors, $\| \sum_{i=1}^{N} X_i \|_{\psi_2}^2$ is the smallest possible number that makes the bound valid. Thus there exists a constant c_0 such that $\| \sum_{i=1}^{N} X_i \|_{\psi_2}^2 \leq c_0 N \, \|X\|_{\psi_2}^2$. On the other hand,

$$\mathbb{E} \exp\left(\lambda X\right) = \left[\mathbb{E} \exp\left(\lambda \sum_{i=1}^{N} X_i\right)\right]^{1/N} \leq \exp\left(\frac{C\lambda^2 \left\| \sum_{i=1}^{N} X_i \right\|_{\psi_2}^2}{N}\right), \forall \lambda \in \mathbb{R}.$$

Analogously, we conclude there exists a constant c_1 such that $\|X\|_{\psi_2}^2 \leq c_1 \| \sum_{i=1}^{N} X_i \|_{\psi_2}^2 / N$. We finish the proof.

(b) $S_N - Np$ can be written as

$$S_N - Np = \sum_{i=1}^{N} (\varepsilon_i - p),$$

where ε_is are i.i.d. Bernoulli(p) random variables. By results in part (a),

$$\|S_N - Np\|_{\psi_2} \asymp \sqrt{N} \, \|\varepsilon_i - p\|_{\psi_2}.$$

The conclusion follows by noting that $\|\varepsilon_i - p\|_{\psi_2} = \sqrt{1/\log(2/p)}$.

2.34 (Subgaussian norm of non-i.i.d. sums)

(a) Find independent, mean zero, subgaussian random variables $X_1, X_2, \ldots$ such that for any N and any coefficient vector $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$, we have

$$\left\| \sum_{i=1}^{N} a_i X_i \right\|_{\psi_2} \asymp \|a\|_\infty$$

where "$\asymp$" indicates equivalence up to absolute constant factors.

(b) Conclude that the inequality in Proposition 2.7.1 cannot be reversed in general.

Solution: We set

$$X_k = \begin{cases} \exp(k/2) & \text{with probability } \exp(-\exp(k))/2, \\ 0 & \text{with probability } 1 - \exp(-\exp(k)), \\ -\exp(k/2) & \text{with probability } \exp(-\exp(k))/2. \end{cases}$$

We have

$$\|X_k\|_{\psi_2} = \sqrt{\frac{\log(\exp(\exp(k)) + 1)}{\exp(k)}} \asymp 1.$$

And for any k, for any $\alpha \in \mathbb{R}$, $\|\alpha X_k\|_{\psi_2} = \alpha \|X_k\|_{\psi_2}$. Suppose $\|a\|_\infty = a_K$. Now we discuss the tail probability of $S_N \coloneqq \sum_{i=1}^{N} a_i X_i$. Let $t = \|a\|_\infty \exp(K/2)$. Then we have

$$\begin{aligned} \mathbb{P}(|S_N| \geq t) &\leq \sum_{i=K}^{N} \mathbb{P}(X_i \neq 0) \\ &= \sum_{i=K}^{N} \exp(-\exp(i)) \\ &\leq \sum_{j=0}^{\infty} [\exp(-\exp(K))]^{\exp(j)} \\ &\leq \sum_{j=0}^{\infty} [\exp(-\exp(K))]^{j} \\ &\leq \frac{\exp(-\exp(K))}{1 - \exp(-\exp(K))} \\ &\leq 2\exp(-\exp(K)) \\ &\leq 2\exp(-t^2/\|a\|_\infty^2). \end{aligned}$$

By Proposition 2.6.6, we have $\|S_N\|_{\psi_2} \lesssim \|a\|_\infty$. On the other hand,

$$\begin{aligned}
\exp(\lambda S_N) =& \mathbb{E} \exp\left[\lambda\left(\|a\|_\infty X_K + \sum_{i\neq k} a_i X_i\right)\right] \\
=& \mathbb{E}\exp\left(\lambda \|a\|_\infty X_K\right) \mathbb{E}\exp\left(\lambda \sum_{i\neq k} a_i X_i\right) \\
\geq& \mathbb{E}\exp\left(\lambda \|a\|_\infty X_K\right) \exp\left(\lambda \sum_{i\neq k} a_i \mathbb{E} X_i\right) \\
=& \mathbb{E}\exp\left(\lambda \|a\|_\infty X_K\right).
\end{aligned}$$

Therefore, one must have $\|S_N\|_{\psi_2} \gtrsim \|a\|_\infty$. We finish the proof.

(b) Trivial.

2.35 (Interpolating between L^1 and L^∞) In Exercise 1.12, we showed how to bound the L^p norm of a random variable X using bounds on its L^1 and L^∞ norms. Now, let us prove a similar result for the subgaussian norm.

(a) Prove that if $\|X\|_{L^1} = a$ and $\|X\|_{L^\infty} = b$, then

$$\|X\|_{\psi_2} \leq \frac{Cb}{\sqrt{\log(2b/a)}}$$

where C is an absolute constant.

(b) Give an example showing that the bound in part (a) is tight (up to an absolute constant) for any a and b.

Solution: (a) First we assume $\|X\|_{L^\infty} = 1$, then $a = \|X\|_{L^1} \leq \|X\|_{L^\infty} = 1$ By results in Exercise 1.12,

$$\|X\|_{L^p} \leq a^{1/p} \leq \frac{100\sqrt{p}}{\sqrt{\log(2/a)}},$$

where the numerical inequality $a^{1/p} \leq 100\sqrt{p}/\sqrt{\log(2/a)}$ holds for any $a \in (0,1)$ and $p \geq 1$. According to Proposition 2.6.6, there exists a constant C such that

$$\|X\|_{\psi_2} \leq \frac{C}{\sqrt{\log(2/a)}}.$$

For general random variable X, $\|X/b\|_{L^\infty} = 1$ and $\|X/b\|_{L^1} = a/b \le 1$. We get

$$\|X/b\|_{\psi_2} \le \frac{C}{\sqrt{\log(2b/a)}}.$$

Thus

$$\|X\|_{\psi_2} \le \frac{Cb}{\sqrt{\log(2b/a)}}.$$

(b) We construct a random variable X such that

$$X = \begin{cases} b & \text{with probability } a/b, \\ 0 & \text{with probability } 1 - a/b. \end{cases}$$

By results in Exercise 2.24,

$$\|X\|_{\psi_2} = \frac{b}{\sqrt{\log(1 + b/a)}} \ge \frac{b}{\sqrt{\log(2b/a)}}.$$

2.36 (Khintchine inequality for $p < 2$) In Theorem 2.7.5, we proved Khintchine inequality for $p \in [2, \infty)$. Let us now extend it to $p \in [1, 2]$.

(a) (**Extrapolation**) Show that any random variable Z with finite L^3 norm satisfies the inequality

$$\|Z\|_{L^2} \le \|Z\|_{L^1}^{1/4} \|Z\|_{L^3}^{3/4}.$$

(b) Let $X_1, \dots, X_N$ be independent subgaussian random variables with zero means and unit variances, and let $a_1, \dots, a_N \in \mathbb{R}$. Prove that

$$cK^{-3} \left(\sum_{i=1}^N a_i^2 \right)^{1/2} \le \mathbb{E} \left| \sum_{i=1}^N a_i X_i \right| \le \left(\sum_{i=1}^N a_i^2 \right)^{1/2}.$$

where $K = \max_i \|X_i\|_{\psi_2}$ and $c > 0$ is an absolute constant.

(c) Conclude that the same inequality holds for the L^p norm of the sum for any $p \in [1, 2]$.

Solution: (a) We have

$$\|Z\|_{L^2} = (\mathbb{E} Z^2)^{1/2}$$

$$
\begin{aligned}
&= (\mathbb{E}|Z|^{1/2} \cdot |Z|^{3/2})^{1/2} \\
&\leq ((\mathbb{E}|Z|)^{1/2} \cdot (\mathbb{E}|Z|^3)^{1/2})^{1/2} \\
&= \|Z\|_{L^1}^{1/4} \|Z\|_{L^3}^{3/4},
\end{aligned}
$$

where the first inequality is due to the Cauchy-Schwarz inequality.

(b) The upper bound is a trivial consequence of the Cauchy-Schwarz inequality. To prove the lower bound, we use results in part(a) to deduce

$$
\left\| \sum_{i=1}^N a_i X_i \right\|_{L^1} \geq \frac{\left\| \sum_{i=1}^N a_i X_i \right\|_{L^2}^4}{\left\| \sum_{i=1}^N a_i X_i \right\|_{L^3}^3}.
$$

The Pythagorean identity and the unit variance assumption give

$$
\left\| \sum_{i=1}^N a_i X_i \right\|_{L^2} = \left(\sum_{i=1}^N a_i^2 \right)^{1/2}.
$$

To bound the denominator, use Proposition 2.7.1 and Proposition 2.6.6,

$$
\begin{aligned}
\left\| \sum_{i=1}^N a_i X_i \right\|_{L^3}^3 \leq & C_0 \left\| \sum_{i=1}^N a_i X_i \right\|_{\psi_2}^3 \\
= & C_0 \left[\left\| \sum_{i=1}^N a_i X_i \right\|_{\psi_2}^2 \right]^{3/2} \\
\leq & C_1 \left[\sum_{i=1}^N a_i^2 \|X_i\|_{\psi_2}^2 \right]^{3/2} \\
\leq & C_1 \left[\sum_{i=1}^N a_i^2 K^2 \right]^{3/2} \\
= & C_1 K^3 \left[\sum_{i=1}^N a_i^2 \right]^{3/2}.
\end{aligned}
$$

Combining the expression for the numerator with the bound for the denominator gives the desired lower bound on $\left\| \sum_{i=1}^N a_i X_i \right\|_{L^1}$.

(c) For any $p \in (1,2)$, it follows from the monotonicity of the L^p norms that

$$cK^{-3}\left(\sum_{i=1}^N a_i^2\right)^{1/2} \le \left\|\sum_{i=1}^N a_i X_i\right\|_{L^1} \le \left\|\sum_{i=1}^N a_i X_i\right\|_{L^p} \le \left\|\sum_{i=1}^N a_i X_i\right\|_{L^2} = \left(\sum_{i=1}^N a_i^2\right)^{1/2}$$

2.37 (A maximal inequality) Prove the following strengthening of Proposition 2.7.6. Let $X_1, X_2, \ldots$ be a sequence of subgaussian random variables, not necessarily independent. Then

$$\left\|\sup_k \frac{X_k}{\sqrt{\log(2k)}}\right\|_{\psi_2} \le C \sup_k \|X_k\|_{\psi_2}$$

where C is an absolute constant.

Solution: We first compute a tail bound for $\sup_k X_k/\sqrt{\log(2k)}$.

$$\begin{aligned}
\mathbb{P}\left(\sup_k \frac{|X_k|}{\sqrt{\log(2k)}} \ge t\right) &\le \sum_k \mathbb{P}\left(\frac{|X_k|}{\sqrt{\log(2k)}} \ge t\right) \\
&\le \sum_k 2\exp\left(\frac{-c\log(2k)t^2}{\|X_k\|_{\psi_2}^2}\right) \\
&\le \sum_k 2\exp\left(\frac{-c\log(2k)t^2}{K^2}\right) \\
&= \sum_k (2k)^{-ct^2/K^2} \\
&\le e^{-ct^2/2K^2} \sum_k k^{-ct^2/K^2},
\end{aligned}$$

where we let $K \coloneqq \sup_k \|X_k\|_{\psi_2}$. As long as $t > \sqrt{2}K/\sqrt{c}$, we have $-ct^2/K^2 < -2$ and $\sum_k k^{-ct^2/K^2} \le C_0$. Therefore, for $t > \sqrt{2}K/\sqrt{c}$,

$$\mathbb{P}\left(\sup_k \frac{|X_k|}{\sqrt{\log(2k)}} \ge t\right) \le C_0 e^{-ct^2/2K^2}.$$

For simplicity of notation, we use Y to denote $\sup_k |X_k|/\sqrt{\log(2k)}$. With the tail bound, we can get

$$\begin{aligned}
\|Y\|_{L^p} &= (\mathbb{E}|Y|^p)^{1/p} \\
&= \left(\int_0^\infty \mathbb{P}(|Y|^p > t)dt\right)^{1/p}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^{\sqrt{2}K/\sqrt{c}} \mathbb{P}(|Y|^p > t) dt + \int_{\sqrt{2}K/\sqrt{c}}^{\infty} \mathbb{P}(|Y|^p > t) dt \right)^{1/p} \\
&\leq \left(\sqrt{2}K/\sqrt{c} + \int_{\sqrt{2}K/\sqrt{c}}^{\infty} \mathbb{P}(|Y| > t^{1/p}) dt \right)^{1/p} \\
&\leq \left(\sqrt{2}K/\sqrt{c} + \int_0^{\infty} e^{-ct^{2/p}/2K^2} dt \right)^{1/p} \\
&= \left(\sqrt{2}K/\sqrt{c} + \frac{p}{2} \left(\frac{\sqrt{2}K}{\sqrt{c}} \right)^p \Gamma\left(\frac{p}{2}\right) \right)^{1/p} \\
&\leq C_1 K \sqrt{p},
\end{aligned}
$$

where the last inequality is by Stirling's formula. We conclude $\left\| \sup_k |X_k| / \sqrt{\log(2k)} \right\|_{\psi_2} \leq C_2 K$ by applying Proposition 2.6.6.

2.38 (Maximum of Gaussians) Let us prove an asymptotically sharp version of the maximal inequality (2.22) for standard normal random variables.

(a) Let $g_1, \ldots, g_N$ be $N(0,1)$ random variables for some $N \geq 2$, that are not necessarily independent. Prove that

$$
\mathbb{E} \max_{i \leq N} g_i \leq \sqrt{2 \ln N} \quad \text{and} \quad \mathbb{E} \max_{i \leq N} |g_i| \leq \sqrt{2 \ln(2N)}.
$$

(b) Prove that if $g_1, g_2, \ldots$ are independent $N(0,1)$, then we have as $N \to \infty$:

$$
\mathbb{E} \max_{i \leq N} g_i = \sqrt{2 \ln N} \, (1 + o(1)) \quad \text{and} \quad \mathbb{E} \max_{i \leq N} |g_i| = \sqrt{2 \ln N} \, (1 + o(1)) \, .
$$

Solution: (a) We apply Jensen's inequality to bound $\exp\left(\lambda \mathbb{E} \max_{i \leq N} g_i\right)$, where λ is any positive constant:

$$
\begin{aligned}
\exp\left(\lambda \mathbb{E} \max_{i \leq N} g_i \right) &\leq \mathbb{E} \left(\exp\left(\lambda \max_{i \leq N} g_i \right) \right) \\
&\leq \mathbb{E} \left(\max_{i \leq N} \exp\left(\lambda g_i\right) \right) \\
&\leq \mathbb{E} \left(\sum_{i \leq N} \exp\left(\lambda g_i\right) \right) \\
&\leq N \mathbb{E} \exp(\lambda g_1)
\end{aligned}
$$

$$= N \exp(\lambda^2/2).$$

Setting $\lambda = \sqrt{2 \log N}$ yields the desired inequality. We can prove $\mathbb{E} \max_{i \le N} |g_i| \le \sqrt{2 \log(2N)}$ with similar techniques.

(b) For any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned}
\mathbb{P}\left(\max_{i \le N} g_i < \sqrt{(1-\epsilon) 2 \log N} \right) &= \prod_{i \le N} \mathbb{P}\left(g_i < \sqrt{(1-\epsilon) 2 \log N} \right) \\
&\le \left(1 - \frac{\sqrt{(1-\varepsilon) 2 \log N}}{(1-\varepsilon) 2 \log N + 1} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{N} \right)^{1-\epsilon} \right)^N .
\end{aligned}$$

When N is sufficiently large,

$$\begin{aligned}
\left(1 - \frac{\sqrt{(1-\varepsilon) 2 \log N}}{(1-\varepsilon) 2 \log N + 1} \frac{1}{\sqrt{2\pi}} \left(\frac{1}{N} \right)^{1-\epsilon} \right)^N &\le \left(1 - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{N} \right)^{1-\epsilon/2} \right)^N \\
&\to 0.
\end{aligned}$$

Therefore, we get

$$\mathbb{P}\left(\max_{i \le N} g_i \ge \sqrt{(1-\epsilon) 2 \log N} \right) \to 1.$$

It follows that

$$\mathbb{E} \max_{i \le N} g_i \ge (1 - o(1)) \sqrt{(1-\epsilon) 2 \log N}.$$

Since ε can be chosen arbitrarily,

$$\mathbb{E} \max_{i \le N} g_i \ge (1 - o(1)) \sqrt{2 \log N}.$$

Also,

$$\mathbb{E} \max_{i \le N} |g_i| \ge \mathbb{E} \max_{i \le N} g_i \ge (1 - o(1)) \sqrt{2 \log N}.$$

We can finish the proof by combining the lower bound with results in part (a).

2.39 (Subgaussian characterization: a maximal inequality) Prove that a random variable X is subgaussian if and only if there exists $K > 0$ such that

$$\mathbb{E} \max_{i \le N} |X_i| \le K \sqrt{\log N} \quad \text{for any } N = 2, 3, \ldots \tag{2.35}$$

where X_i are independent copies of X. We have already proved half of this statement: Proposition 2.7.6 shows that if X is subgaussian, then (2.35) holds with $K \leq C\|X\|_{\psi_2}$. Now prove the converse: if (2.35) holds, then X is subgaussian and $\|X\|_{\psi_2} \leq CK$.

Solution: Since X_i s are independent copies of X,

$$\mathbb{P}(\max_{i\leq N} |X_i| < t) = \prod_{i\leq N} \mathbb{P}(|X_i| < t) = (1 - \mathbb{P}(|X| \geq t))^N.$$

On the other hand, by (2.35),

$$\mathbb{P}(\max_{i\leq N} |X_i| < t) \geq 1 - \frac{K\sqrt{\log N}}{t}.$$

Setting $t = C_0 K\sqrt{\log N}$, where $C_0 \geq 1$ and $N = 2, 3, \dots$ yields

$$(1 - \mathbb{P}(|X| \geq C_0 K\sqrt{\log N}))^N \geq 1 - \frac{1}{C_0},$$

for any $N = 2, 3, \dots$. We can conclude that there exists constant C_1 such that

$$\mathbb{P}(|X| \geq C_0 K\sqrt{\log N}) \leq \frac{C_1}{N}.$$

In other words, for any $t' \geq C_0 K\sqrt{\log 2}$,

$$\mathbb{P}(|X| \geq t') \leq \frac{C_1}{\lfloor \exp(t'^2/C_0^2 K^2) \rfloor} \leq 2\exp(-t'^2/C_2^2 K^2).$$

Observe that for any $t' < C_0 K\sqrt{\log 2}$,

$$\mathbb{P}(|X| \geq t') \leq 1 \leq 2\exp(-t'^2/C_0^2 K^2).$$

Therefore, for any $t' \in [0, \infty)$,

$$\mathbb{P}(|X| \geq t') \leq 2\exp(-t'^2/C_3^2 K^2),$$

where $C_3 = \max\{C_0, C_2\}$. We can conclude that $\|X\|_{\psi_2} \leq C'K$ using Proposition 2.6.2.

2.40 (A surgeon's view: the exact subgaussian norm) You may have noticed that most results involving the subgaussian norm hold up to an absolute constant factor rather than exactly.

To refine these results, we can redefine the subgaussian norm using the equivalent property (iv) from Proposition 2.6.1. For motivation, recall that the MGF of a normal random variable $X \sim N(\mu, \sigma^2)$ satisfies $\mathbb{E}e^{\lambda(X-\mu)} = e^{\sigma^2\lambda^2/2}$ for all $\lambda \in \mathbb{R}$. Inspired by this, we can define the *subgaussian variance* of a random variable X as

$$\mathrm{Var}_G(X) := \inf\left\{\sigma^2 : \mathbb{E}e^{\lambda(X-\mathbb{E}X)} \leq e^{\sigma^2\lambda^2/2} \text{ for all } \lambda \in \mathbb{R}\right\}.$$

Recall that the L^2 norm of X can be expressed as $\|X\|_{L^2}^2 = \mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}X)^2$. Inspired by this, define the *exact subgaussian norm* of a random variable X by the identity

$$\|X\|_G^2 := \mathrm{Var}_G(X) + (\mathbb{E}X)^2.$$

Prove the following.

(a) The exact subgaussian norm indeed defines a norm on the space of subgaussian random variables.

(b) The exact and standard subgaussian norms are equivalent up to absolute constant factors: $c_1\|X\|_{\psi_2} \leq \|X\|_G \leq c_2\|X\|_{\psi_2}$.

(c) We have $\mathrm{Var}(X) \leq \mathrm{Var}_G(X)$ and $\|X\|_{L^2} \leq \|X\|_G$, with equalities holding if X is normally distributed.

(d) If $X_1, \ldots, X_N$ are independent and mean zero, then

$$\mathrm{Var}_G\left(\sum_{i=1}^N X_i\right) \leq \sum_{i=1}^N \mathrm{Var}_G(X_i).$$

Rewriting this as $\left\|\sum_{i=1}^N X_i\right\|_G^2 \leq \sum_{i=1}^N \|X_i\|_G^2$ gives an exact version of Proposition 2.7.1.

(e) An exact version of centering (Lemma 2.7.8): $\|X - \mathbb{E}X\|_G \leq \|X\|_G$.

Solution: (a) First, it is straightforward to verify that for any subgaussian r.v. X, $0 \leq \|X\|_G < \infty$. And $\|X\|_G = 0$ iff $X = 0$ almost surely. For any $s \neq 0$, we have

$$\begin{aligned}\mathrm{Var}_G(sX) &= \inf\left\{\sigma^2 : \mathbb{E}e^{\lambda(sX-\mathbb{E}sX)} \leq e^{\sigma^2\lambda^2/2} \text{ for all } \lambda \in \mathbb{R}\right\} \\ &= \inf\left\{\sigma^2 : \mathbb{E}e^{(s\lambda)(X-\mathbb{E}X)} \leq e^{\sigma^2(s\lambda)^2/2s^2} \text{ for all } \lambda \in \mathbb{R}\right\}\end{aligned}$$

$$= s^2 \operatorname{Var}_G(X).$$

Thus

$$\|sX\|_G = \sqrt{s^2(\operatorname{Var}_G(X) + (\mathbb{E}X)^2)} = |s| \, \|X\|_G .$$

To prove the triangular inequality, we first assume X, Y are mean-zero subgaussian random variables. For $p, q \geq 1$ such that $1/p + 1/q = 1$,

$$\begin{aligned}
\mathbb{E} \exp \lambda(X+Y) &\leq (\mathbb{E} \exp(\lambda p X))^{1/p} (\mathbb{E} \exp(\lambda q Y))^{1/q} \\
&\leq (\exp(p^2 \operatorname{Var}_G(X) \lambda^2/2))^{1/p} (\exp(q^2 \operatorname{Var}_G(Y) \lambda^2/2))^{1/q} \\
&= \exp((p \operatorname{Var}_G(X) + q \operatorname{Var}_G(Y)) \lambda^2/2).
\end{aligned}$$

We can deduce that $\operatorname{Var}_G(X+Y) \leq p \operatorname{Var}_G(X) + q \operatorname{Var}_G(Y)$. Setting $p = \sqrt{\operatorname{Var}_G(Y)/\operatorname{Var}_G(X)} + 1$ yields

$$\operatorname{Var}_G(X+Y) \leq \left(\sqrt{\operatorname{Var}_G(X)} + \sqrt{\operatorname{Var}_G(Y)} \right)^2 .$$

Observing that for any $c \in \mathbb{R}$,

$$\operatorname{Var}_G(X+c) = \operatorname{Var}_G(X),$$

for general subgaussian random variables X and Y we also have

$$\operatorname{Var}_G(X+Y) \leq \left(\sqrt{\operatorname{Var}_G(X)} + \sqrt{\operatorname{Var}_G(Y)} \right)^2 .$$

Therefore,

$$\begin{aligned}
\|X+Y\|_G &= \left\| \left(\sqrt{\operatorname{Var}_G(X+Y)}, \mathbb{E}(X+Y) \right) \right\|_2 \\
&\leq \left\| \left(\sqrt{\operatorname{Var}_G(X)} + \sqrt{\operatorname{Var}_G(Y)}, \mathbb{E}(X+Y) \right) \right\|_2 \\
&= \left\| \left(\sqrt{\operatorname{Var}_G(X)}, \mathbb{E}X \right) + \left(\sqrt{\operatorname{Var}_G(Y)}, \mathbb{E}Y \right) \right\|_2 \\
&\leq \left\| \left(\sqrt{\operatorname{Var}_G(X)}, \mathbb{E}X \right) \right\|_2 + \left\| \left(\sqrt{\operatorname{Var}_G(Y)}, \mathbb{E}Y \right) \right\|_2 \\
&\leq \|X\|_G + \|Y\|_G .
\end{aligned}$$

We finish the proof.

(b) By Proposition 2.6.2 and Lemma 2.7.8,

$$\mathrm{Var}_G(X) \lesssim \|X - \mathbb{E}X\|_{\psi_2}^2 \lesssim \|X\|_{\psi_2}^2 ,$$

and

$$(\mathbb{E}X)^2 \leq \mathbb{E}X^2 \lesssim \|X\|_{\psi_2}^2 .$$

Thus

$$\|X\|_G^2 \lesssim \|X\|_{\psi_2}^2 .$$

On the other hand,

$$\begin{aligned}
\|X\|_{\psi_2}^2 &\leq \Big(\|X - \mathbb{E}X\|_{\psi_2} + \|\mathbb{E}X\|_{\psi_2}\Big)^2 \\
&\lesssim \|X - \mathbb{E}X\|_{\psi_2}^2 + \|\mathbb{E}X\|_{\psi_2}^2 \\
&\lesssim \|X - \mathbb{E}X\|_{\psi_2}^2 + \mathbb{E}X^2,
\end{aligned}$$

The second inequality holds due to Cauchy's inequality, and the last inequality holds due to results in Exercise 2.24. The conclusion follows noting that $\|X - \mathbb{E}X\|_{\psi_2}^2 \lesssim \mathrm{Var}_G(X)$ by Proposition 2.7.8.

(c) Performing Taylor expansion around $\lambda = 0$ gives

$$\mathbb{E}e^{\lambda(X-\mathbb{E}X)} = 1 + \frac{\mathrm{Var}(X)\lambda^2}{2} + o(\lambda^2).$$

Therefore, for any σ_0^2 satisfying $\mathbb{E}e^{\lambda(X-\mathbb{E}X)} \leq e^{\sigma_0^2\lambda^2/2}$, $\forall\lambda \in \mathbb{R}$, we must have $\sigma_0^2 \geq \mathrm{Var}(X)$. The remaining arguments are straightforward.

(d) We observe that

$$\begin{aligned}
\mathbb{E}e^{\lambda\sum_{i=1}^N X_i} &= \mathbb{E}\prod_{i=1}^N e^{\lambda X_i} \\
&= \prod_{i=1}^N \mathbb{E}e^{\lambda X_i} \\
&\leq \prod_{i=1}^N \mathbb{E}e^{\mathrm{Var}_G(X_i)\lambda^2/2} \\
&= \mathbb{E}e^{\sum_{i=1}^N \mathrm{Var}_G(X_i)\lambda^2/2}.
\end{aligned}$$

We complete the proof.

(e) The argument holds trivially since $\mathrm{Var}_G(X - \mathbb{E}X) = \mathrm{Var}_G(X)$.

2.42 (A bird's eye view: Orlicz norms) Here is a general framework that covers most norms we have seen so far. Let $\psi : [0, \infty) \to [0, \infty)$ be a nondecreasing, convex function with $\psi(0) = 0$. The Orlicz norm of a random variable X is defined as

$$\|X\|_\psi = \inf \left\{ K > 0 : \mathbb{E}\psi\left(\frac{|X|}{K}\right) \leq 1 \right\}.$$

(a) Verify that this indeed defines a norm on the set of all random variables (on the same probability space) for which $\|X\|_\psi$ is finite.

(b) Explain why the following are examples of Orlicz norms: the L^p norm for any $p \in [1, \infty)$, the subgaussian norm $\|X\|_{\psi_2}$, and the subexponential norm $\|X\|_{\psi_1}$.

Solution: (a) It is straightforward to verify that for any random variable X satisfying $\|X\|_\psi < 0$ we have: 1) $\|X\|_\psi \geq 0$ with $\|X\|_\psi = 0$ iff $X = 0$ a.s.; 2) for any $\alpha \in \mathbb{R}$, $\|\alpha X\|_\psi = |\alpha| \|X\|_\psi$. Now it suffices to verify for any random variables X, Y, $\|X + Y\|_\psi \leq \|X\|_\psi + \|Y\|_\psi$. We have

$$\begin{aligned}
&\mathbb{E}\psi\left(\frac{|X+Y|}{\|X\|_\psi + \|Y\|_\psi}\right) \\
&\leq \mathbb{E}\psi\left(\frac{|X|+|Y|}{\|X\|_\psi + \|Y\|_\psi}\right) \quad (\psi \text{ is non-decreasing}) \\
&\leq \mathbb{E}\psi\left(\frac{\|X\|_\psi \cdot (|X|/\|X\|_\psi) + \|Y\|_\psi \cdot (|Y|/\|Y\|_\psi)}{\|X\|_\psi + \|Y\|_\psi}\right) \\
&\leq \frac{\|X\|_\psi}{\|X\|_\psi + \|Y\|_\psi}\mathbb{E}\psi\left(\frac{|X|}{\|X\|_\psi}\right) + \frac{\|Y\|_\psi}{\|X\|_\psi + \|Y\|_\psi}\mathbb{E}\psi\left(\frac{|Y|}{\|Y\|_\psi}\right) \quad (\psi \text{ is convex}) \\
&\leq 1.
\end{aligned}$$

Thus it must hold that $\|X + Y\|_\psi \leq \|X\|_\psi + \|Y\|_\psi$.

(b) The L_p norm is a special case of Oclicz norms where $\psi(x) = x^p$, the sub-gaussian norm is a special case of Oclicz norms where $\psi(x) = e^{x^2} - 1$, and the sub-exponential norm is a special case of Oclicz norms where $\psi(x) = e^x - 1$.

2.46 (Subexponential Khintchine inequality) Prove the following subexponential version of Theorem 2.7.5. Let $X_1, \dots, X_N$ be independent, mean zero, subexponential random variables, and $a = (a_1, \dots, a_N) \in \mathbb{R}^N$. Then, for every $p \in [2, \infty)$, we have

$$\left\| \sum_{i=1}^N a_i X_i \right\|_{L^p} \le CK \left(\sqrt{p} \|a\|_2 + p \|a\|_\infty \right)$$

where $K = \max_i \|X_i\|_{\psi_1}$ and C is an absolute constant.

Solution: By Corollary 2.9.2, for every $t \ge 0$,

$$\begin{aligned} \mathbb{P}\left\{ \left| \sum_{i=1}^N a_i X_i \right| \ge t \right\} &\le 2 \exp\left[-c \min\left(\frac{t^2}{K^2 \|a\|_2^2}, \frac{t}{K\|a\|_\infty} \right) \right] \\ &\le 2 \exp\left(\frac{-ct^2}{K^2 \|a\|_2^2} \right) + 2 \exp\left(\frac{-ct}{K\|a\|_\infty} \right). \end{aligned}$$

where $K = \max_i \|X_i\|_{\psi_1}$. Therefore,

$$\begin{aligned} \left\| \sum_{i=1}^N a_i X_i \right\|_{L^p}^p &= \int_0^\infty \mathbb{P}\left(\left| \sum_{i=1}^N a_i X_i \right| > t \right) p t^{p-1} dt \\ &\le 2p \int t^{p-1} \exp\left(\frac{-ct^2}{K^2 \|a\|_2^2} \right) dt + 2p \int t^{p-1} \exp\left(\frac{-ct}{K\|a\|_\infty} \right) dt \\ &\le 2\Gamma\left(\frac{p}{2} + 1 \right) \left(\frac{K^2 \|a\|_2^2}{c} \right)^{\frac{p}{2}} + 2\Gamma(p+1) \left(\frac{K\|a\|_\infty}{c} \right)^p \\ &\le 6 \left(\frac{p+1}{2} \right)^{\frac{p+1}{2}} \left(\frac{K^2 \|a\|_2^2}{c} \right)^{\frac{p}{2}} + 6(p+1)^{p+1} \left(\frac{K\|a\|_\infty}{c} \right)^p, \end{aligned}$$

where we use the numerical inequality $\Gamma(x) \le 3x^x$ in the last inequality. Taking the p-th root of both sides gives the conclusion.

2.47 (Bernstein inequality for bounded distributions)

(a) Let X be a mean zero random variable satisfying $|X| \le K$ a.s. Prove the following bound on the MGF of X:

$$\mathbb{E} \exp(\lambda X) \le \exp\left(g(\lambda) \mathbb{E} X^2 \right) \quad \text{where} \quad g(\lambda) = \frac{\lambda^2/2}{1 - |\lambda| K/3},$$

provided that $|\lambda| < 3/K$.

(b) Deduce Theorem 2.9.5 by the exponential moment method.

Solution: (a) For $|z| < 3$, we have

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k|} \le 1 + z + \frac{z^2}{2} \sum_{k=0}^{\infty} \frac{|z|^k}{(k+2)!/2} \le 1 + z + \frac{z^2}{2} \sum_{k=0}^{\infty} \frac{|z|^k}{3^k} = 1 + z + \frac{z^2/2}{1 - |z|/3}.$$

Assume $|\lambda| < 3/K$,

$$\mathbb{E} \exp(\lambda X) \le 1 + \mathbb{E} \frac{\lambda^2 X^2/2}{1 - \lambda|X|/3} \le \exp(\mathbb{E} X^2 g(\lambda)).$$

(b) We have

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \ge t \right) \le e^{-\lambda t} \prod_{i=1}^{N} e^{\lambda X_i} \le \exp\left(-\lambda t + g(\lambda)\sigma^2 \right).$$

Substituting $\lambda = \frac{t}{\sigma^2 + Kt/3}$ gives

$$\mathbb{P}\left(\sum_{i=1}^{N} X_i \ge t \right) \le \exp\left(-\frac{t^2/2}{\sigma^2 Kt/3} \right).$$

Repeating the argument for $-X_i$, we obtain the same bound for $\mathbb{P}(\sum_{i=1}^{N} \le -t)$. A combination of the two bounds completes the proof.

2.48 (Bennett inequality) Let us prove the following strengthening of Bernstein inequality (Theorem 2.9.5). Let $X_1, \ldots, X_N$ be independent, mean zero random variables satisfying $|X_i| \le K$ all i. Then, for every $t \ge 0$, we have

$$\mathbb{P}\left\{ \left| \sum_{i=1}^{N} X_i \right| \ge t \right\} \le 2 \exp\left[-\frac{\sigma^2}{K^2} h\left(\frac{Kt}{\sigma^2} \right) \right] \tag{2.36}$$

where $\sigma^2 = \sum_{i=1}^{N} \mathbb{E} X_i^2$ and $h(u) = (1+u)\ln(1+u) - u$. Prove this result as follows:

(a) Let X be a mean zero random variable satisfying $|X| \le K$. Show that the MGF of X satisfies the following bound for any $\lambda > 0$:

$$\mathbb{E} \exp(\lambda X) \le \exp\left[\frac{\mathbb{E} X^2}{K^2} \left(e^{\lambda K} - 1 - \lambda K \right) \right].$$

(b) Use the exponential moment method to deduce (2.36).

(c) Check that in the small deviations range, where $u = Kt/\sigma^2 \approx 0$, we have $h(u) \approx u^2/2$, so Bennett inequality gives approximately the Gaussian tail bound $\exp(-t^2/\sigma^2)$.

(d) Check that we always have $h(u) \geq \frac{1}{2} u \ln u$, so Bennett inequality gives a Poisson tail bound $2(\sigma^2/Kt)^{t/2K}$ in the large deviations range, similar to Exercise 2.13.

Solution: (a) Expanding the MGF of X using its Taylor series gives

$$\begin{aligned}
\mathbb{E}\exp(\lambda X) &= \sum_{k=0}^{\infty} \frac{\lambda^k \mathbb{E}X^k}{k!} \\
&= 1 + \sum_{k=2}^{\infty} \frac{\lambda^k \mathbb{E}X^k}{k!} \\
&\leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k K^{k-2} \mathbb{E}X^2}{k!} \\
&= 1 + \frac{\mathbb{E}X^2}{K^2} \sum_{k=2}^{\infty} \frac{\lambda^k K^k \mathbb{E}X^2}{k!} \\
&= 1 + (e^{\lambda K} - 1 - \lambda K) \\
&\leq \exp\left[\frac{\mathbb{E}X^2}{K^2} (e^{\lambda K} - 1 - \lambda K) \right].
\end{aligned}$$

(b) With the exponential moment method, let $S_N = \sum_{i=1}^N X_i$ for simplification, we have

$$\begin{aligned}
\mathbb{P}(S \geq t) &\leq \exp(-\lambda t) \prod_{i=1}^{N} \exp(\lambda X_i) \\
&\leq \exp\left[\frac{\sigma^2}{K^2} (e^{\lambda K} - 1 - \lambda K) - \lambda t \right].
\end{aligned}$$

We minimize the exponent with respect to λ to derive the tightest bound. Define $f(\lambda) = \sigma^2(e^{\lambda K} - 1 - \lambda K)/K^2 - \lambda t$. $f(\lambda)$ is convex and $f'(\lambda) = \sigma^2 e^{\lambda K}/K - (t + \sigma^2/K)$. Thus the minimum of $f(\lambda)$ is attained at $\lambda^* = \log(tK/\sigma^2 + 1)/K$. Substituting λ^* back into the exponent, we get

$$\begin{aligned}
\mathbb{P}(S \geq t) &\leq \exp(-\lambda t) \prod_{i=1}^{N} \exp(\lambda X_i) \\
&\leq \exp\left[-\frac{\sigma^2}{K^2} h(Kt/\sigma^2) \right].
\end{aligned}$$

The result for $|\sum X_i|$ follows from applying the same argument to $-S_N$ and using a union bound.

(c) Trivial.

(d) Trivial.

3 Random Vectors in High Dimensions

3.2 (Thin shell, generalized) Let us relax the assumptions in Exercise 3.1 and prove the variance bound for any distribution with a bounded fourth moment. Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent coordinates X_i that satisfy $\mathbb{E}X_i^2 = 1$ and $\mathbb{E}X_i^4 \leq K^4$. Show that

$$\operatorname{Var}(\|X\|_2) \leq K^4 \quad \text{and} \quad \sqrt{n} - \frac{K^4}{\sqrt{n}} \leq \mathbb{E}\|X\|_2 \leq \sqrt{n}.$$

Solution: We first establish the variance bound. By the property that $\operatorname{Var}(Y) \leq \mathbb{E}[(Y-c)^2]$ for any constant c, we have

$$\begin{aligned}\mathbb{E}(\|X\|_2 - \sqrt{n})^2 &= \mathbb{E}\left(\frac{\|X\|_2^2 - n}{\|X\|_2 + \sqrt{n}}\right)^2 \\ &\leq \frac{\mathbb{E}(\|X\|_2^2 - n)^2}{n}\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{E}(\|X\|_2^2 - n)^2 &= \mathbb{E}\,\|X\|_2^4 + n^2 - 2n\mathbb{E}\,\|X\|_2^2 \\ &= \sum_{i\neq j} \mathbb{E}X_i^2 X_j^2 + \sum_i \mathbb{E}X_i^4 + n^2 - 2n\mathbb{E}\,\|X\|_2^2 \\ &= nK^4 + n(n-1) + n^2 - 2n^2 \\ &\leq nK^4.\end{aligned}$$

Substituting this back, we obtain

$$\operatorname{Var}(\|X\|_2) \leq \frac{n(K^4-1)}{n} = K^4 - 1 \leq K^4.$$

Next, we bound the expectation. For the upper bound, we use Jensen's inequality:

$$\mathbb{E}\|X\|_2 = \mathbb{E}\left[\sqrt{\|X\|_2^2}\right] \leq \sqrt{\mathbb{E}[\|X\|_2^2]} = \sqrt{n}.$$

The lower bound hold since

$$\mathbb{E}(\|X\|_2 - \sqrt{n})^2 = \mathbb{E}\,\|X\|_2^2 + n - 2\sqrt{n}\mathbb{E}\,\|X\|_2 = 2n - 2\sqrt{n}\mathbb{E}\,\|X\|_2 \leq K^4.$$

Rearranging terms yields $\mathbb{E}\,\|X\|_2 \geq \sqrt{n} - K^4/2\sqrt{n}$. This completes the proof.

3.3 (Thin shell, reversed) Let us show reverse bounds in Exercise 3.2. Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with independent coordinates X_i that satisfy $\mathbb{E}X_i^2 = 1$, $\mathrm{Var}(X_i^2) > \alpha$ and $\mathbb{E}X_i^6 \le \beta$ for some $\alpha, \beta > 0$. Prove that if n is large enough (depending on α and β) then

$$\mathrm{Var}\left(\|X\|_2\right) \ge c\alpha \quad \text{and} \quad \mathbb{E}\|X\|_2 \le \sqrt{n} - \frac{c\alpha}{\sqrt{n}}.$$

Explain why a lower bound on the variance is an essential assumption.

Solution: Let $Z = \frac{\|X\|_2^2}{n}$, then $\mathbb{E}\|X\|_2 = \sqrt{n}\,\mathbb{E}[\sqrt{Z}]$. Since $\mathbb{E}Z = \frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i^2 = 1$, let us expand $f(z) = \sqrt{z}$ around $z = 1$. By Taylor's theorem, there exists a random C between Z and 1 such that

$$\sqrt{Z} = 1 + \tfrac{1}{2}(Z-1) - \tfrac{1}{8}(Z-1)^2 + \tfrac{1}{16}C^{-5/2}(Z-1)^3.$$

Taking expectations,

$$\mathbb{E}\sqrt{Z} = 1 - \tfrac{1}{8}\mathrm{Var}(Z) + R_3, \qquad R_3 := \tfrac{1}{16}\,\mathbb{E}\Big[C^{-5/2}(Z-1)^3\Big].$$

Let $Y_i = X_i^2 - 1$, so that $Z - 1 = \frac{1}{n}\sum_{i=1}^n Y_i$. Then

$$\mathrm{Var}(Z) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^n Y_i\right) = \frac{1}{n^2}\sum_{i=1}^n \mathrm{Var}(Y_i) \ge \frac{\alpha}{n}.$$

The last inequality is by the assumption $\mathrm{Var}(X_i^2) = \mathrm{Var}(Y_i) \ge \alpha$. Next, We show $R_3 = o(n^{-1})$. First,

$$|R_3| \le \tfrac{1}{16}\,\mathbb{E}\Big[|C|^{-5/2}|Z-1|^3\Big].$$

As Z convereges to 1 in probability, so there exists a constant K such that

$$\mathbb{E}\Big[|C|^{-5/2}|Z-1|^3\Big] \le K\,\mathbb{E}|Z-1|^3.$$

Recall that $Z - 1 = \frac{1}{n}\sum_{i=1}^n Y_i$, $\mathbb{E}|Z-1|^3 = \frac{1}{n^3}\,\mathbb{E}|\sum_{i=1}^n Y_i|^3$. By the Rosenthal–Burkholder inequality for independent mean-zero variables,

$$\mathbb{E}\left|\sum_{i=1}^n Y_i\right|^3 \le C_p\left(\sum_{i=1}^n \mathbb{E}|Y_i|^3 + \left(\sum_{i=1}^n \mathbb{E}Y_i^2\right)^{3/2}\right).$$

By the triangle inequality, $|X_i^2 - 1|^3 \le (|X_i^2| + 1)^3$. Applying the numerical inequality $(a+b)^p \le$

$2^{p-1}(a^p + b^p)$ for $a, b \geq 0$ and $p \geq 1$:

$$(|X_i^2| + 1)^3 \leq 2^{3-1}(|X_i^2|^3 + 1^3) = 4(X_i^6 + 1)$$

Taking the expectation:

$$\mathbb{E}|Y_i|^3 \leq 4(\mathbb{E}X_i^6 + 1) \leq 4\beta + 1 =: C_1$$

Also,

$$\mathbb{E}Y_i^2 = \mathrm{Var}(X_i^2) = \mathbb{E}[(X_i^2)^2] - (\mathbb{E}X_i^2)^2 = \mathbb{E}X_i^4 - 1$$

By Lyapunov's inequality, $(\mathbb{E}X_i^4)^{1/4} \leq (\mathbb{E}X_i^6)^{1/6}$, we have $\mathbb{E}X_i^4 \leq \beta^{2/3}$. Thus $\mathbb{E}Y_i^2 \leq \beta^{2/3} - 1 =: C_2$. Combining these two bounds gives

$$\mathbb{E}\left|\sum_{i=1}^{n} Y_i\right|^3 \leq K\big(nC_1 + (nC_2)^{3/2}\big) = O(n^{3/2}),$$

and therefore $|R_3| = O(n^{-3/2})$. We prove that there exists constant c such that

$$\mathbb{E}\|X\|_2 \leq \sqrt{n} - \frac{c\alpha}{\sqrt{n}}.$$

Next, we observe that

$$\mathrm{Var}(\|X\|_2) = n\big(1 - (\mathbb{E}\sqrt{Z})^2\big) = n(1 - \mathbb{E}\sqrt{Z})(1 + \mathbb{E}\sqrt{Z}).$$

Since

$$1 - \mathbb{E}\sqrt{Z} = \tfrac{1}{8}\mathrm{Var}(Z) - R_3 \geq \frac{c\alpha}{n},$$

it follows that

$$\mathrm{Var}(\|X\|_2) \geq n(1 - \mathbb{E}\sqrt{Z}) \geq c\alpha.$$

The strictly positive variance ensures that the concavity of $\sqrt{z}$ introduces a strictly negative $\Theta(n^{-1})$ term in the Taylor expansions.

3.4 (PCA maximizes the explained variance) Let X be a mean zero random vector X in $\mathbb{R}^n$ whose covariance matrix has eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and eigenvectors $v_1, \ldots, v_n$.

(a) Prove that the orthogonal projection P_k onto the span of $v_1, \ldots, v_k$ satisfies

$$\mathbb{E}\|P_k X\|_2^2 = \sum_{i=1}^{k} \mathrm{Var}\left(\langle X, v_i \rangle\right) = \sum_{i=1}^{k} \lambda_i.$$

(b) Prove that any rank-k orthogonal projection P in $\mathbb{R}^n$ satisfies

$$\mathbb{E}\|PX\|_2^2 \leq \mathbb{E}\|P_k X\|_2^2.$$

Interpret these results as follows: PCA maximizes the explained variance of the data, with the explained fraction given by s_k/s_n, where $s_k = \lambda_1 + \ldots + \lambda_k$.

Solution: (a) WLOG, we assume $v_1, ..., v_n$ forms an standard orthogonal basis of $\mathbb{R}^n$. By definition of P_k, we have $P_k = \sum_{i=1}^{k} v_i v_i^\top$. Therefore,

$$\begin{aligned}
\mathbb{E}\left\|P_k X\right\|_2^2 &= \mathbb{E}\left\|\sum_{i=1}^{k} v_i v_i^\top X\right\|_2^2 \\
&= \sum_{i=1}^{k} \mathbb{E}\left\|v_i v_i^\top X\right\|_2^2 \\
&= \sum_{i=1}^{k} \mathbb{E}(v_i^\top X)^2 \left\|v_i\right\|_2^2 \\
&= \sum_{i=1}^{k} \mathbb{E}(v_i^\top X)^2 \\
&= \sum_{i=1}^{k} \mathrm{Var}(\langle v_i, X \rangle) \\
&= \sum_{i=1}^{k} v_i^\top \Sigma v_i \\
&= \sum_{i=1}^{k} \lambda_i,
\end{aligned}$$

where the second equality uses the orthogonality of $\{v_i\}_{i=1}^k$, the fourth equality uses $\|v_i\|_2 = 1$, and the second to last equality is by Proposition 3.2.1.

(b) The orthogonal projection operator P can be expressed as $P = \sum_{i=1}^{k} u_i u_i^\top$, where $\{u_i\}_{i=1}^k$

are orthonormal. With similar logic as in the proof of (a), $\mathbb{E}\,\|PX\|_2^2 = \sum_{i=1}^k u_i^\top \Sigma u_i$. Hence

$$\begin{aligned}
\mathbb{E}\,\|PX\|_2^2 &= \sum_{i=1}^k u_i^\top \Sigma u_i \\
&= \sum_{i=1}^k u_i^\top \left(\sum_{j=1}^n \lambda_j v_j v_j^\top \right) u_i \\
&= \sum_{j=1}^n \sum_{i=1}^k \lambda_j \langle u_i, v_j \rangle^2
\end{aligned}$$

Let $a_j \coloneqq \sum_{i=1}^k \langle u_i, v_j \rangle^2$, by Bessel's inequality, we have $a_j \le \|v_j\|_2^2 = 1$. Also,

$$\begin{aligned}
\sum_{j=1}^n a_j &= \sum_{j=1}^n \sum_{i=1}^k \langle u_i, v_j \rangle^2 \\
&= \sum_{i=1}^k \sum_{j=1}^n \langle u_i, v_j \rangle^2 \\
&= \sum_{i=1}^k \sum_{j=1}^n \|u_i\|_2^2 \\
&= k.
\end{aligned}$$

Since $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, we can get

$$\mathbb{E}\,\|PX\|_2^2 = \sum_{j=1}^n \lambda_j a_j \le \sum_{j=1}^k \lambda_j.$$

3.5 (Expected ℓ^p norm of a random vector) Let $n \in \mathbb{N}$ and $p \in [1, \infty]$. Consider a random vector $X = (X_1, \ldots, X_n)$ whose coordinates are independent subgaussian random variables. Show that

$$\mathbb{E}\|X\|_p \le \begin{cases} CK\sqrt{p}\, n^{1/p}, & p \le \log n \\ CK\sqrt{\log n}, & p \ge \log n \end{cases}$$

where C is an absolute constant and $K = \max_i \|X_i\|_{\psi_2}$.

Solution: By Jensen's inequality and Proposition 2.6.6, we can get

$$\mathbb{E}\,\|X\|_p \le \left(\sum_{i=1}^n \mathbb{E}|X_i|^p \right)^{1/p} \le \left(\sum_{i=1}^n \left(C_0 \|X_i\|_{\psi_2} \sqrt{p} \right)^p \right)^{1/p} \le C_1 K \sqrt{p}\, n^{1/p}.$$

When $p \geq \log n$, we have

$$\mathbb{E}\,\|X\|_p \leq \mathbb{E}\,\|X\|_{\lfloor \log n \rfloor} \leq C_1 K \sqrt{\log n}\, n^{1/\lfloor \log n \rfloor} \leq C_2 K \sqrt{n}.$$

We complete the proof.

3.6 (Expected ℓ^p norm of a Gaussian vector) Show that the bounds in Exercise **3.5** can be reversed for the standard gaussian random vector. Specifically, let $n \in \mathbb{N}$ and $p \in [1, \infty]$. Consider a random vector $X = (X_1, \ldots, X_n)$ whose coordinates are independent $N(0,1)$ random variables. Show that

$$\mathbb{E}\|X\|_p \geq \begin{cases} c\sqrt{p}\, n^{1/p}, & p \leq \log n \\ c\sqrt{\log n}, & p \geq \log n \end{cases}$$

where $c > 0$ is an absolute constant.

Solution: When $p \geq \sqrt{\log n}$, by results in Exercise 2.38 we have $\mathbb{E}\,\|X\|_p \geq \mathbb{E}\,\|X\|_\infty \geq c\sqrt{\log n}$. When $p \leq \log n$, we split the n coordinates uniformly into $N = \lceil n/e^p \rceil$ groups, say, $A_1, \ldots, A_N$. Each group contains at least $c_0 e^p$ coordinates, where $c_0 < 1$ is a constant. Therefore, we can get

$$\begin{aligned} \mathbb{E}\,\|X\|_p &= \mathbb{E}\left(\sum_{i=1}^n X_i^p\right)^{1/p} \\ &\geq \mathbb{E}\left(\sum_{j=1}^N \left(\sup_{i \in A_j} X_i\right)^p\right)^{1/p} \\ &\geq \left(\sum_{j=1}^N \left(\mathbb{E}\sup_{i \in A_j} X_i\right)^p\right)^{1/p} \\ &\geq \left(\sum_{j=1}^N (c_1\sqrt{p})^p\right)^{1/p} \\ &= N^{1/p} c_1 \sqrt{p} \\ &\geq c n^{1/p} \sqrt{p}. \end{aligned}$$

Here the second inequality holds due to Exercise 1.14, and the third inequality holds by Exercise 2.38. We finish the proof.

3.7 (Small ball probability) Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent coordinates with continuous distributions, whose densities are all bounded by K.

(a) Let us show that X is very unlikely to land in any given ball of radius $o(\sqrt{n})$. Specifically, for any fixed $a \in \mathbb{R}^n$ and $\varepsilon > 0$, prove that

$$\mathbb{P}\{\|X - a\|_2 \le \varepsilon\sqrt{n}\} \le \left(\sqrt{2\pi e} K \varepsilon\right)^n.$$

Will see in Exercises 4.28–4.29 that this is asymptotically sharp for small ε.

(b) Deduce for $n \ge 2$ that

$$\mathbb{E}\left[\frac{1}{\|X\|_2}\right] \le \frac{CK}{\sqrt{n}}.$$

Explain why for $n = 1$ the expected value can be infinite.

Solution: WLOG we can assume $a = 0$. For any $\lambda > 0$,

$$\begin{aligned}
\mathbb{P}(\|X\|_2 \le \varepsilon\sqrt{n}) &= \mathbb{P}(\|X\|_2^2 \le \varepsilon^2 n) \\
&= \mathbb{P}(-\lambda^2 \|X\|_2^2 / \varepsilon^2 \ge -\lambda^2 n) \\
&= \mathbb{P}\left(\exp\left\{-\lambda^2 \|X\|_2^2 / \varepsilon^2\right\} \ge \exp\left\{-\lambda^2 n\right\}\right) \\
&\le \exp(\lambda^2 n)\mathbb{E}\exp\left\{-\lambda^2 \|X\|_2^2 / \varepsilon^2\right\} \\
&\le \exp(\lambda^2 n)\prod_{i=1}^n \mathbb{E}\exp\left\{-\lambda^2 X_i^2 / \varepsilon^2\right\}.
\end{aligned}$$

Next, we bound the expectation term. Let the probability density function of X_1 be $f(x)$, which is bounded by a constant K. The random variable $Y = \lambda X_1/\varepsilon$ has a density $g(y) = (\varepsilon/\lambda)\, f((\varepsilon/\lambda)\, y)$, which is thus bounded by εK. We have

$$\mathbb{E}\exp\left\{-\lambda^2 X_1^2/\varepsilon^2\right\} = \int_0^\infty e^{-y^2} g(y) dy \le (\varepsilon K/\lambda)\int_{-\infty}^\infty e^{-y^2} dy = \sqrt{\pi}\varepsilon K/\lambda.$$

Similarly, for $i = 2, \ldots, n$, we also have $\mathbb{E}\exp\left\{-\lambda^2 X_1^2/\varepsilon^2\right\} \le \sqrt{\pi}\varepsilon K/\lambda$. Substituting this into our main inequality gives

$$\mathbb{P}(\|X\|_2 \le \varepsilon\sqrt{n}) \le \left(\sqrt{\pi}\varepsilon K \exp(\lambda^2)/\lambda\right)^n.$$

Setting $\lambda = 1/\sqrt{2}$,

$$\mathbb{P}(\|X\|_2 \le \varepsilon\sqrt{n}) \le \left(\sqrt{2\pi e}K\varepsilon\right)^n.$$

We complete the proof.

(b) We observe that

$$\begin{aligned}\mathbb{E}\left[\frac{\sqrt{n}}{K\|X\|_2}\right] &= \int_0^\infty \mathbb{P}\left(\frac{\sqrt{n}}{K\|X\|_2} > x\right) dx \\ &= \int_0^\infty \mathbb{P}\left(\|X\|_2 < \frac{\sqrt{n}}{Kx}\right) dx.\end{aligned}$$

By results in (a), $\mathbb{P}\left(\|X\|_2 < \frac{\sqrt{n}}{Kx}\right) \le (C_0/x)^n$. Therefore, for $n > 2$,

$$\mathbb{E}\left[\frac{\sqrt{n}}{K\|X\|_2}\right] \le \int_0^\infty 1 \wedge (C_0/x)^n dx \le C_0 + \int_{C_0}^\infty (C_0/x)^2 \le C.$$

It follows that

$$\mathbb{E}\left[\frac{1}{\|X\|_2}\right] \le CK\sqrt{n}.$$

For $n = 1$, we simply consider the random varible $X \sim \mathrm{Unif}([0, 1/K])$. It is straightforward to verify $\mathbb{E}[1/X] = +\infty$.

3.8 (Expectation of an isotropic random vector)

(a) Prove that any isotropic random vector X in $\mathbb{R}^n$ satisfies $\|\mathbb{E}X\|_2 \le 1$.

(b) Demonstrate that this bound is optimal in general: for each $n \in \mathbb{N}$, find an isotropic random vector X in $\mathbb{R}^n$ that satisfies $\|\mathbb{E}X\|_2 = 1$.

Solution: (a) Let $\mu \coloneqq \mathbb{E}X$ and $U = \langle \mu/\|\mu\|_2, X\rangle$. Then we have

$$\mathbb{E}\,U = \frac{\mu^\top \mu}{\|\mu\|_2} = \|\mu\|_2, \ \mathbb{E}\,U^2 = \frac{\mu^\top}{\|\mu\|_2}\mathbb{E}XX^\top \frac{\mu}{\|\mu\|_2} = 1.$$

Since $\mathbb{E}\,U^2 \ge (\mathbb{E}\,U)^2$, it must holds that $\|\mu\|_2^2 \le 1$.

(b) Let $X \sim \mathrm{Unif}\left(\{\sqrt{n}e_1, \ldots, \sqrt{n}e_n\}\right)$. It is straight forward to verify $\|\mathbb{E}X\| = 1$.

3.10 (The standard score of a random vector)

(a) Let $\mu \in \mathbb{R}^n$ be a fixed vector, Σ be a fixed $n \times n$ symmetric positive semidefinite matrix, and Z be a mean zero, isotropic random vector in $\mathbb{R}^n$. Check that the random vector $X = \mu + \Sigma^{1/2} Z$ has mean μ and covariance matrix Σ.

(b) Conversely, let X be a random vector with mean μ and covariance matrix Σ. Find a mean zero, isotropic random vector Z that satisfies $X = \mu + \Sigma^{1/2} Z$ almost surely.

Solution: (a) Trivial. Without loss of generality, assume $\mu = 0$. If Σ is invertible, define $Z = \Sigma^{-1/2} X$; then $\Sigma^{1/2} Z = X$ as required. If Σ is singular, let its eigen-decomposition be

$$\Sigma = \sum_{i=1}^{n} \lambda_i v_i v_i^\top,$$

where $\lambda_i > 0$ for $i \le r$ and $\lambda_i = 0$ for $i > r$. Let Y be any mean-zero isotropic random vector, and define

$$Z = \left(\sum_{i \le r} \lambda_i^{-1/2} v_i v_i^\top \right) X + \left(\sum_{i > r} v_i v_i^\top \right) Y.$$

Since $\{v_i\}$ form an orthonormal basis, we have

$$\begin{aligned} \Sigma^{1/2} Z &= \left[\sum_{i \le r} \lambda_i^{1/2} v_i v_i^\top \right] \left[\sum_{j \le r} \lambda_j^{-1/2} v_j v_j^\top X + \sum_{j > r} v_j v_j^\top Y \right] \\ &= \sum_{i \le r} \lambda_i^{1/2} v_i v_i^\top \lambda_i^{-1/2} v_i v_i^\top X \\ &= \sum_{i \le r} v_i v_i^\top X. \end{aligned}$$

Finally, note that $\sum_{i \le r} v_i v_i^\top$ is the orthogonal projection onto $\mathrm{Im}(\Sigma) = \mathrm{span}\{v_1, \ldots, v_r\}$. Since X has covariance Σ, it lies almost surely in this subspace, and hence $\sum_{i \le r} v_i v_i^\top X = X$. Therefore, $\Sigma^{1/2} Z = X$ as desired.

3.11 (A randomly permuted vector) Let X be a random vector obtained by randomly permuting the coordinates of a given vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 + \cdots + x_n = 0$ and $x_1^2 + \cdots + x_n^2 = 1$. Specifically, let σ be a random permutation of $\{1, \ldots, n\}$ chosen uniformly from the set of all $n!$ permutations, and set $X_i = x_{\sigma(i)}$ for each $i = 1, \ldots, n$.

(a) Compute the mean and covariance matrix of X.

(b) You will find that the coordinates of X are negatively correlated. Can you heuristically explain why?

Solution: Since each coordinate X_i is uniformly distributed over $\{x_1, \ldots, x_n\}$,

$$\mathbb{E}X_i = \frac{1}{n}\sum_{k=1}^{n} x_k = 0, \qquad \text{and hence} \qquad \mathbb{E}X = 0.$$

Moreover,

$$\mathbb{E}[X_i^2] = \frac{1}{n}\sum_{k=1}^{n} x_k^2 = \frac{1}{n}.$$

For $i \neq j$, the pair (X_i, X_j) corresponds to sampling two distinct elements without replacement from $\{x_1, \ldots, x_n\}$. Therefore,

$$\mathbb{E}[X_i X_j] = \frac{1}{n(n-1)}\sum_{p \neq q} x_p x_q.$$

Using $\sum_{p=1}^{n} x_p = 0$ and $\sum_{p=1}^{n} x_p^2 = 1$, we find

$$\sum_{p \neq q} x_p x_q = \left(\sum_{p=1}^{n} x_p\right)^2 - \sum_{p=1}^{n} x_p^2 = -1.$$

Hence,

$$\mathbb{E}[X_i X_j] = -\frac{1}{n(n-1)}.$$

Thus, the covariance matrix of X is

$$\mathrm{Cov}(X) = \frac{1}{n} I_n - \frac{1}{n(n-1)}(\mathbf{1}\mathbf{1}^\top - I_n) = \frac{1}{n-1}\left(I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^\top\right),$$

where $\mathbf{1} = (1, \ldots, 1)^\top$.

(b) Because the coordinates of X are obtained by permuting a fixed zero-sum vector, a large value in one coordinate forces the other coordinates to be smaller on average, so that the total remains zero.

3.13 (Maximum norm of random vectors) Let's prove is a high-dimensional version of (2.22).

(a) Let $X_1, \ldots, X_N \in \mathbb{R}^n$ be random vectors, not necessarily independent. Assume that the coordinates X_{ij} of each random vector X_i are independent, subgaussian, and satisfy $\mathbb{E}X_{ij}^2 = 1$.

Prove that

$$\mathbb{E} \max_{i \le N} \|X_i\|_2 \le \sqrt{n} + CK^2 \sqrt{\log N}$$

where $K = \max_{ij} \|X_{ij}\|_{\psi_2}$.

(b) Show that this bound is tight for i.i.d. standard normal random vectors. Specifically, if $X_1, \dots, X_N \sim N(0, I_n)$ are independent, prove that

$$\mathbb{E} \max_{i \le N} \|X_i\|_2 \ge c \left(\sqrt{n} + \sqrt{\log N} \right).$$

Solution: (a) By Theorem 3.1.1, for each $i \le N$,

$$\big\| \, \|X_i\|_2 - \sqrt{n} \, \big\|_{\psi_2} \le CK^2.$$

Hence, the random variables $\|X_i\|_2 - \sqrt{n}$ are subgaussian with subgaussian norm bounded by CK^2. Applying Proposition 2.7.6 (which bounds the expected maximum of subgaussian random variables), we obtain

$$\mathbb{E} \max_{i \le N} \big(\|X_i\|_2 - \sqrt{n} \big) \le CK^2 \sqrt{\log N}.$$

Rearranging terms yields

$$\mathbb{E} \max_{i \le N} \|X_i\|_2 \le \sqrt{n} + CK^2 \sqrt{\log N},$$

as desired.

(b) From Exercise 3.2, we know that

$$\mathbb{E} \|X_1\|_2 \simeq \sqrt{n}.$$

Moreover, by Exercise 2.38, the maximum norm among N i.i.d. standard Gaussian vectors satisfies

$$\mathbb{E} \max_{i \le N} \|X_i\|_2 \succeq \sqrt{\log N}.$$

Combining these two estimates gives

$$\mathbb{E} \max_{i \le N} \|X_i\|_2 \ge c\big(\sqrt{n} + \sqrt{\log N} \big),$$

which shows that the bound in part (a) is sharp up to absolute constants.

3.17 (Normal, uncorrelated, but dependent) Find normal random variables X and Y that are uncorrelated but not independent.

Solution: We simply set $X \sim N(0,1)$ and $Y = \varepsilon X$, where ε is a Rademacher random variable.

3.18 (Ginibre random matrices) A basic model for random matrices is the *Ginibre ensemble*. A Ginibre random matrix is an $n \times n$ matrix G with independent $N(0,1)$ entries. Prove that the Ginibre distribution is invariant under multiplication by orthogonal matrices, that is, for any fixed $n \times n$ orthogonal matrix U, both UG and GU have the same distribution as G.

Solution: We consider the 1D marginal of UG. For any $A \in \mathbb{R}^{n \times n}$,

$$\mathbb{E}\operatorname{vec}(A)^\top \operatorname{vec}(UG) = 0,$$

and

$$\begin{aligned}
\operatorname{Var}(\operatorname{vec}(A)^\top \operatorname{vec}(UG)) &= \mathbb{E}[\operatorname{vec}(A)^\top \operatorname{vec}(UG)]^2 \\
&= \operatorname{vec}(A)^\top \mathbb{E}\left[\operatorname{vec}(UG)\operatorname{vec}(UG)^\top\right] \operatorname{vec}(A) \\
&= \operatorname{vec}(A)^\top (I_n \otimes U) \mathbb{E}\left[\operatorname{vec}(G)\operatorname{vec}(G)^\top\right] (I_n \otimes U) \operatorname{vec}(A) \\
&= \operatorname{vec}(A)^\top (I_n \otimes U)(I_n \otimes U)^\top \operatorname{vec}(A) \\
&= \operatorname{vec}(A)^\top \operatorname{vec}(A).
\end{aligned}$$

We can conclude every 1D marginal of UG has the same distribution as G, thus UG has the same distribution as G. Similarly, we can prove GU has the same distribution as G.

3.19 (GOE random matrices) A basic model for symmetric random matrices is the *Gaussian orthogonal ensemble* (GOE). A GOE random matrix is an $n \times n$ symmetric matrix A with independent normal entries on and above the diagonal, where the diagonal entries are $\mathcal{N}(0,2)$ and the off-diagonal entries are $\mathcal{N}(0,1)$.

(a) Check that A has the same distribution as $(G+G^\top)/\sqrt{2}$ where G is a Ginibre random matrix from Exercise 3.18. This explains the different scaling of the diagonal and off-diagonal entries of A in the definition.

(b) Prove that the GOE distribution is invariant under orthogonal conjugation, that is, for any fixed $n \times n$ orthogonal matrix U, the matrix $UAU^\top$ has the same distribution as A. This

explains the name "Gaussian orthogonal(-invariant) ensemble".

Solution: (a) Trivial.

(b) Let G be a Ginibre random matrix, combining results in part (a) and Exercise 3.18 gives

$$\begin{aligned} UAU^\top &\overset{d}{=} \frac{UGU^\top + UG^\top U^\top}{\sqrt{2}} \\ &\overset{d}{=} \frac{GU^\top + UG^\top}{\sqrt{2}} \\ &\overset{d}{=} \frac{G + G^\top}{\sqrt{2}} \\ &\overset{d}{=} A, \end{aligned}$$

where we use $\overset{d}{=}$ to denote equality in distribution.

3.23 (Many random points are in convex position) A finite set of points in $\mathbb{R}^n$ is in *convex position* if no point lies in the convex hull of the others (see Figure 3.11). Let $N \le e^{cn}$. Following the steps below, prove that with probability at least $1 - e^{-cn}$, independent Gaussian random vectors $g_1, \dots, g_N \sim N(0, I_n)$ are in convex position. Here, as usual, $c > 0$ is an absolute constant of your choice. Surprisingly, this shows that exponentially many random points are in convex position!

(a) Show that, with high probability, g_1 is linearly separated from the other points, i.e. there exists $x \in \mathbb{R}^n$ (possibly random) satisfying $\langle g_1, x\rangle > \max_{j=2,\dots,N} \langle g_j, x\rangle$.

(b) Deduce that, with high probability, g_1 is not in the convex hull of the other points.

(c) Conclude by the union bound.

Solution: (a) Let $x \coloneqq g_1/\|g_1\|_2$. Using Theorem 3.1.1, there exists a constant $C_0 > 0$ such that, with probability at least $1 - e^{-C_0 n}$, we have $\langle g_1, x\rangle = \|g_1\|_2 > \frac{\sqrt{n}}{2}$. On the other hand, for any $j \neq 1$, we have $\langle x, g_j\rangle \sim N(0,1)$, and these random variables are independent. Therefore,

$$\mathbb{P}\left(\max_{j=2,\dots,N} \langle g_j, x\rangle > \frac{\sqrt{n}}{2}\right) = 1 - \prod_{j=2}^{N} \mathbb{P}\left(\langle g_j, x\rangle \le \frac{\sqrt{n}}{2}\right).$$

Since $\langle g_j, x\rangle \sim N(0,1)$, we have $\mathbb{P}\left(\langle g_j, x\rangle \le \frac{\sqrt{n}}{2}\right) \ge 1 - e^{-\frac{n}{2}}/\sqrt{n}$. Thus,

$$\mathbb{P}\left(\max_{j=2,\dots,N} \langle g_j, x\rangle > \frac{\sqrt{n}}{2}\right) \le 1 - \left(1 - e^{-\frac{n}{2}}/\sqrt{n}\right)^{e^{cn}} \le \frac{e^{(c-\frac{1}{2})n}}{\sqrt{n}}.$$

As long as $c < \frac{1}{2}$, there exists a constant $C_1 > 0$ such that $\max_{j=2,\dots,N}\langle g_j, x\rangle < \langle g_1, x\rangle$ with probability at least $1 - e^{-C_1 n}$.

(b) Suppose g_1 is in the convex hull of $g_2, \dots, g_N$. Then there exist coefficients $\alpha_2, \dots, \alpha_N \geq 0$ such that

$$g_1 = \sum_{j=2}^{N} \alpha_j g_j, \quad \sum_{j=2}^{N} \alpha_j \leq 1.$$

For any $x \in \mathbb{R}^n$, we have

$$\langle g_1, x\rangle = \sum_{j=2}^{N} \alpha_j \langle g_j, x\rangle.$$

This implies

$$\max_{j=2,\dots,N} \langle g_j, x\rangle \geq \langle g_1, x\rangle.$$

Since we know that $\max_{j=2,\dots,N}\langle g_j, x\rangle < \langle g_1, x\rangle$ with probability at least $1 - e^{-C_1 n}$, we conclude that g_1 is not in the convex hull of the other points with probability at least $1 - e^{-C_1 n}$.

(c) To finish the proof, apply the union bound to obtain the probability that any point g_k is in the convex hull of the others. We have:

$$\begin{aligned} &\mathbb{P}\left(\exists k, g_k \text{ is in the convex hull of the other points}\right) \\ \leq &\sum_{k=1}^{N} \mathbb{P}\left(g_k \text{ is in the convex hull of the other points}\right) \leq N e^{-C_1 n}. \end{aligned}$$

Since $N \leq e^{cn}$, we obtain

$$\mathbb{P}\left(\exists k, g_k \text{ is in the convex hull of the other points}\right) \leq e^{cn} e^{-C_1 n} = e^{-(C_1 - c)n}.$$

Thus, with probability at least $1 - e^{-(C_1 - c)n}$, no point g_k lies in the convex hull of the others. The conclusion follows as long as $2c < C_1$. From (a), we require $C_1 < \frac{1}{2} - c$, so any choice of $c \in (0, \frac{1}{6})$ works.

3.24 (Polar decomposition of the uniform distribution on a ball) Let X be uniformly distributed on the unit Euclidean ball centered at the origin in $\mathbb{R}^n$. Decompose X as

$$X = rZ$$

where $Z \sim \text{Unif}(S^{n-1})$ and r is a random variable in $[0,1]$ with density nx^{n-1}, and r and Z are independent.

Solution: Let X be uniformly distributed on the Euclidean unit ball

$$B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}.$$

Write $X = rZ$ with $r = \|X\| \in [0,1]$ and $Z = X/\|X\| \in S^{n-1}$ (define Z arbitrarily at $X = 0$). Recall the spherical-coordinate formula for Lebesgue integration: for any nonnegative or integrable function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{B^n} g(x)\,dx = \int_0^1 \int_{S^{n-1}} g(r\omega)\, r^{\,n-1}\, d\sigma(\omega)\, dr,$$

where $d\sigma$ denotes the surface measure on S^{n-1}. In particular, the Jacobian contributes the factor r^{n-1}.

Since X is uniformly distributed on B^n, its density with respect to Lebesgue measure is the constant $1/\operatorname{Vol}(B^n)$. For any measurable $A \subseteq S^{n-1}$ and $0 \le a < b \le 1$,

$$\begin{aligned}\mathbb{P}\big(Z \in A,\ a \le r \le b\big) &= \frac{1}{\operatorname{Vol}(B^n)} \int_{\{x \in B^n :\, a \le \|x\| \le b,\ x/\|x\| \in A\}} dx \\ &= \frac{1}{\operatorname{Vol}(B^n)} \int_a^b \int_A r^{\,n-1}\, d\sigma(\omega)\, dr.\end{aligned}$$

Hence the joint distribution of (r, Z) has density (with respect to $d\sigma\, dr$) $\frac{r^{\,n-1}}{\operatorname{Vol}(B^n)}$.

Integrating out r yields the marginal of Z:

$$\mathbb{P}(Z \in A) = \frac{1}{\operatorname{Vol}(B^n)} \int_0^1 \int_A r^{\,n-1}\, d\sigma(\omega)\, dr = \frac{1}{\operatorname{Vol}(B^n)} \Big(\int_0^1 r^{\,n-1} dr \Big) \sigma(A) = \frac{\sigma(A)}{n\ \operatorname{Vol}(B^n)}.$$

The well-known relation between volume and surface measure for the unit ball is $\operatorname{Vol}(B^n) = \sigma(S^{n-1})/n$, so

$$\mathbb{P}(Z \in A) = \frac{\sigma(A)}{\sigma(S^{n-1})},$$

showing that $Z \sim \text{Unif}(S^{n-1})$.

Similarly, integrating out $\omega \in S^{n-1}$ yields the marginal distribution of r:

$$\mathbb{P}(a \le r \le b) = \frac{1}{\operatorname{Vol}(B^n)} \int_a^b \Big(\int_{S^{n-1}} r^{\,n-1} d\sigma(\omega) \Big) dr = \frac{\sigma(S^{n-1})}{\operatorname{Vol}(B^n)} \int_a^b r^{\,n-1} dr.$$

Using $\mathrm{Vol}(B^n) = \frac{1}{n}\sigma(S^{n-1})$ we obtain

$$\mathbb{P}(a \le r \le b) = \int_a^b n r^{n-1}\, dr,$$

so r has density $f_r(t) = nt^{n-1}$ on $[0,1]$.

The factorization of the joint density into a product of the marginals implies that r and Z are independent. This completes the proof.

3.26 (Delocalization of random vector on the sphere) The most "spread out" or "delocalized" unit vector x in $\mathbb{R}^n$ has all coordinates with the same magnitude, i.e. $|x_i| = 1/\sqrt{n}$ for all i. It turns out that a random vector is almost delocalized. Show that a random vector X uniformly distributed on the unit Euclidean sphere in $\mathbb{R}^n$ satisfies

$$\mathbb{E}\|X\|_\infty \asymp \sqrt{\frac{\log n}{n}}.$$

As always, "$\asymp$" means equivalence up to absolute constant factors.

Solution: Let $g \sim N(0, I_n)$. Then

$$X \overset{d}{=} \frac{g}{\|g\|_2}, \qquad \|X\|_\infty \overset{d}{=} \frac{\|g\|_\infty}{\|g\|_2}.$$

Thus it suffices to bound $\mathbb{E}(\|g\|_\infty / \|g\|_2)$.

Upper bound. By Exercise 3.7, for any $t > 0$,

$$\mathbb{P}\left(\frac{1}{\|g\|_2} > \sqrt{\frac{t}{n}}\right) = \mathbb{P}\left(\|g\|_2 < \sqrt{\frac{n}{t}}\right) \le \left(\frac{e}{t}\right)^{n/2}.$$

Moreover, Proposition 2.7.6 yields

$$\mathbb{P}\Big(\|g\|_\infty > \sqrt{t \log n}\Big) \le 2e^{-C_0 t}.$$

Therefore, for every $t > 0$,

$$\mathbb{P}\left(\frac{\|g\|_\infty / \|g\|_2}{\sqrt{\log n / n}} > t\right) \le \left(\frac{e}{t}\right)^{n/2} + 2e^{-C_0 t}.$$

Integrating this tail bound,

$$\mathbb{E}\left(\frac{\|g\|_\infty/\|g\|_2}{\sqrt{\log n/n}}\right) = \int_0^\infty \mathbb{P}\left(\frac{\|g\|_\infty/\|g\|_2}{\sqrt{\log n/n}} > t\right) dt$$
$$\leq \int_0^\infty \min\left\{1,\, \left(\frac{e}{t}\right)^{n/2} + 2e^{-C_0 t}\right\} dt.$$

For $n \geq 4$,

$$\int_e^\infty \left(\frac{e}{t}\right)^{n/2} dt \leq \int_e^\infty (e/t)^2\, dt < \infty, \qquad \int_e^\infty 2e^{-C_0 t} dt < \infty.$$

Hence

$$\mathbb{E}\left(\frac{\|g\|_\infty/\|g\|_2}{\sqrt{\log n/n}}\right) \preceq 1,$$

which gives the desired upper bound.

Lower bound. Let $A = \{\|g\|_2 < 2\sqrt{n}\}$; by Theorem 3.1.1,

$$\mathbb{P}(A) \geq 1 - 2e^{-C_2 n}.$$

Then

$$\mathbb{E}\left(\frac{\|g\|_\infty}{\|g\|_2}\right) \geq \mathbb{E}\left(\frac{\|g\|_\infty}{\|g\|_2}\mathbf{1}_A\right) \geq \frac{\mathbb{E}(\|g\|_\infty \mathbf{1}_A)}{2\sqrt{n}}.$$

Define $B = \{\|g\|_\infty > \frac{1}{2}\sqrt{\log n}\}$. Since the coordinates are independent,

$$\mathbb{P}(B) = 1 - \mathbb{P}\big(|g_i| \leq \tfrac{1}{2}\sqrt{\log n}\ \forall i\big)$$
$$= 1 - \prod_{i=1}^n \left(1 - \mathbb{P}\left(|g_i| > \tfrac{1}{2}\sqrt{\log n}\right)\right).$$

Using Proposition 2.1.2 and the bound $1 + x \leq e^x$,

$$\mathbb{P}\left(|g_i| > \tfrac{1}{2}\sqrt{\log n}\right) \geq \frac{\frac{1}{2}\sqrt{\log n}}{\frac{1}{2}(\log n) + 2} \cdot \frac{1}{\sqrt{2\pi}}\, n^{-1/8}, \qquad \mathbb{P}(B) \geq 1 - \exp\left(-\frac{n^{7/8}}{3\sqrt{\log n}}\right).$$

Since $A \cap B \subseteq A$,

$$\mathbb{E}\left(\frac{\|g\|_\infty}{\|g\|_2}\right) \geq \frac{\mathbb{E}(\|g\|_\infty \mathbf{1}_{A\cap B})}{2\sqrt{n}}$$
$$\geq \frac{\sqrt{\log n}/2}{2\sqrt{n}}\, \mathbb{P}(A \cap B)$$

$$\geq \sqrt{\frac{\log n}{16n}} \left(1 - \mathbb{P}(A^c) - \mathbb{P}(B^c)\right)$$
$$\geq \sqrt{\frac{\log n}{16n}} \left(1 - 2e^{-C_2 n} - e^{-n^{7/8}/(3\sqrt{\log n})}\right)$$
$$\succeq \sqrt{\frac{\log n}{n}}.$$

Combining the upper and lower bounds completes the proof.

3.27 (1D marginals of the sphere) Let X and Y be random vectors uniformly distributed on the Euclidean unit sphere and the Euclidean unit ball in $\mathbb{R}^n$.

(a) Prove that the density of any 1D marginal of Y is proportional to $(1-x^2)^{\frac{n-1}{2}}$ for $x \in [-1,1]$.

(b) Prove that the density of any 1D marginal of X is proportional to $(1-x^2)^{\frac{n-3}{2}}$ for $x \in [-1,1]$. Surprisingly, the 1D marginals of a 3D sphere are uniformly distributed!

Solution: (a) By symmetry, it suffices to consider the density of Y_1. The marginal density $f_{Y_1}(x)$ at $x \in [-1,1]$ is proportional to the $(n-1)$-dimensional volume of the intersection of the unit ball B^n with the hyperplane $H_x = \{y \in \mathbb{R}^n : y_1 = x\}$.

For a fixed $y_1 = x$, the condition $\sum_{i=1}^n y_i^2 \leq 1$ becomes:

$$\sum_{i=2}^n y_i^2 \leq 1 - x^2$$

This describes an $(n-1)$-dimensional ball of radius $r = \sqrt{1-x^2}$. The volume of this ball is proportional to r^{n-1}. Therefore:

$$f_{Y_1}(x) \propto (\sqrt{1-x^2})^{n-1} = (1-x^2)^{\frac{n-1}{2}}.$$

(b) Let $Z = (Z_1, \ldots, Z_n)$ be a standard normal random vector, i.e., $Z \sim N(0, I_n)$. It is a standard result that$X \stackrel{d}{=} Z/\|Z\|_2$. Consequently, the first coordinate X_1 can be represented as:

$$X_1 = \frac{Z_1}{\sqrt{Z_1^2 + Z_2^2 + \cdots + Z_n^2}} = \frac{Z_1}{\sqrt{Z_1^2 + S}},$$

where $S = \sum_{i=2}^n Z_i^2$. Since Z_i are independent standard normal variables, $Z_1^2 \sim \chi_1^2$, $S \sim \chi_{n-1}^2$, and they are independent. To find the density of X_1, we first consider the squared random variable

$W = X_1^2$. Suppose $U \sim \chi_a^2$ and $V \sim \chi_b^2$ are independent, the ratio $\frac{U}{U+V}$ follows a Beta distribution $Beta(a/2, b/2)$. Thus,

$$W \sim \text{Beta}\left(\frac{1}{2}, \frac{n-1}{2}\right).$$

The probability density function of W is supported on $(0, 1)$ and satisfies:

$$f_W(w) \propto w^{\frac{1}{2}-1}(1-w)^{\frac{n-1}{2}-1} = w^{-\frac{1}{2}}(1-w)^{\frac{n-3}{2}}.$$

Now we perform the change of variables to find the density of X_1. Since $W = X_1^2$, for $x \in (-1, 1) \setminus \{0\}$, we have $w = x^2$ and $|\frac{dw}{dx}| = 2|x|$. Using the transformation formula:

$$\begin{aligned} f_{X_1}(x) &= f_W(x^2)\left|\frac{dw}{dx}\right| \\ &\propto (x^2)^{-\frac{1}{2}}(1-x^2)^{\frac{n-3}{2}} \cdot 2|x| \\ &= \frac{1}{|x|}(1-x^2)^{\frac{n-3}{2}} \cdot 2|x| \\ &\propto (1-x^2)^{\frac{n-3}{2}}. \end{aligned}$$

3.28 (Large cube probability) Let us establish a surprising fact: a standard normal random vector in $\mathbb{R}^n$ is likely to be very close to a cube with constant side length, yet it is highly unlikely to land in a cube even a hundred times larger!

(a) Let π_a denote a *metric projection* onto the cube $[-a, a]^n$, which returns a point $y = \pi_a(x)$ in the cube closest to the input point x in the Euclidean distance. Show that the metric projection can be obtained by *soft thresholding* the coordinates of x as follows: $y_i = x_i$ if $|x_i| \le a$ and $y_i = a\,\text{sign}(x_i)$ otherwise, for each $i = 1, \ldots, n$.

(b) Show that there exists an absolute constants n_0 and a such that for every $n > n_0$, a random vector $g \sim N(0, I_n)$ satisfies the following with probability at least 0.99:

$$\text{dist}\,(g, [-a, a]^n) < 0.01\|g\|_2 \quad \text{and} \quad g \notin [-100a, 100a]^n.$$

Here $\text{dist}(x, T) = \inf_{y \in T} \|x - y\|_2$ denotes the distance from a point x to a set T.

Solution: (a) For any $x \in \mathbb{R}^n$ and $y \in [-a,a]^n$,

$$\|y-x\|_2^2 = \sum_{i=1}^n (y_i - x_i)^2,$$

so the minimization over the cube separates coordinatewise. The conclusion follows in a straightforward manner.

(b) Write

$$\operatorname{dist}(g,[-a,a]^n)^2 = \sum_{i=1}^n X_i, \qquad X_i := \mathbf{1}\{|g_i| > a\}(|g_i|-a)^2.$$

Since $(|g_i|-a)^2\mathbf{1}\{|g_i| > a\} \to 0$ is pointwise and is dominated by $(|g_i|+a)^2$, the dominated convergence theorem implies $\mathbb{E}X_i \to 0$ as $a \to \infty$. Choose a large enough that $\mathbb{E}X_i \le 10^{-8}$. Then

$$\mathbb{E}\left[\operatorname{dist}(g,[-a,a]^n)^2\right] = n\mathbb{E}X_i \le 10^{-8}n,$$

and Markov's inequality yields $\operatorname{dist}(g,[-a,a]^n) \le 10^{-2}\sqrt{n}$ with probability at least $1-10^{-3}$.

Standard concentration gives $\|g\|_2 \ge 0.7\sqrt{n}$ with probability at least $1-10^{-3}$ for all sufficiently large n. Since $\mathbb{P}(g \in [-100a,100a]^n) = \big(\mathbb{P}(|g_1| \le 100a)\big)^n$ decays exponentially in n. For sufficiently large n, we also have $\mathbb{P}(g \in [-100a,100a]^n) \le 10^{-3}$.

Combining these results, we conclude that for sufficient large a and n,

$$\mathbb{P}\left(\operatorname{dist}(g,[-a,a]^n) < 0.01\|g\|_2 \text{ and } g \notin [-100a,100a]^n\right) \ge 0.99.$$

3.30 (The Mercedes-Benz frame) Consider N equispaced points on a circle of radius $\sqrt{2/N}$ (see Figure 3.7 for $N=3$). Show that they form a Parseval frame in $\mathbb{R}^2$.

Solution: Let

$$u_k = \sqrt{\frac{2}{N}}\big(\cos(2\pi k/N),\, \sin(2\pi k/N)\big), \qquad k = 0,\dots,N-1.$$

It suffices to verify for any $v \in \mathbb{R}^2$, $v = \sum_{k=0}^{N-1}\langle u_k, v\rangle u_k$. For $v=(x,y)\in\mathbb{R}^2$, we compute

$$\sum_{k=0}^{N-1}\langle u_k, v\rangle u_k = \frac{2}{N}\sum_{k=0}^{N-1}\begin{pmatrix} x\cos^2(2\pi k/N) + y\sin(2\pi k/N)\cos(2\pi k/N) \\ x\cos(2\pi k/N)\sin(2\pi k/N) + y\sin^2(2\pi k/N)\end{pmatrix}.$$

Using the identities

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta), \quad \sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta), \quad \sin\theta\cos\theta = \tfrac{1}{2}\sin 2\theta,$$

each component reduces to sums of $\cos(4\pi k/N)$ and $\sin(4\pi k/N)$. Since $N \geq 3$, the complex geometric sum

$$\sum_{k=0}^{N-1} e^{4\pi i k/N} = 0$$

implies

$$\sum_{k=0}^{N-1} \cos(4\pi k/N) = 0, \quad \sum_{k=0}^{N-1} \sin(4\pi k/N) = 0.$$

Therefore,

$$\sum_{k=0}^{N-1} \cos^2(2\pi k/N) = \frac{N}{2}, \quad \sum_{k=0}^{N-1} \sin^2(2\pi k/N) = \frac{N}{2}, \quad \sum_{k=0}^{N-1} \sin(2\pi k/N)\cos(2\pi k/N) = 0.$$

Plugging these into the previous expression gives the conclusion.

3.33 (Subgaussian rotation invariance) Proposition 3.3.1 showed that the standard normal distribution is rotation invariant. Subgaussian distributions are not necessarily rotation invariant, but the subgaussian norm is. For any orthogonal $n \times n$ matrix U and subgaussian vector $X \in \mathbb{R}^n$, show that UX is also subgaussian, and

$$\|UX\|_{\psi_2} = \|X\|_{\psi_2}.$$

Solution: For any orthogonal $n \times n$ matrix U,

$$\|UX\|_{\psi_2} = \sup_{\|x\|_2=1} \langle UX, x\rangle = \sup_{\|x\|_2=1} \langle X, U^\top x\rangle$$

Noting that for $\|x\|_2 = 1$, $\|Ux\|_2 = 1$. We get

$$\|UX\|_{\psi_2} = \sup_{\|x\|_2=1} \langle X, U^\top x\rangle \leq \sup_{\|x\|_2=1} \langle X, x\rangle = \|X\|_{\psi_2}.$$

As $X = U^\top(UX)$, we also have $\|X\|_{\psi_2} \leq \|UX\|_{\psi_2}$. We finish the proof.

3.34 (Subgaussian quadratic forms) Let A be an $m \times n$ random matrix with independent, mean-zero, subgaussian rows A_i. For any fixed $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, show that $u^\top Av$ is a subgaussian random variable, and

$$\|u^\top Av\|_{\psi_2} \le C\|u\|_2\|v\|_2 \max_i \|A_i\|_{\psi_2}.$$

Explain why the result still holds if the same assumptions are made about the *columns* of A instead of the rows.

Solution: It suffices to show that for $u \in S^{m-1}$ and $v \in S^{n-1}$, $\left\|u^\top Av\right\|_{\psi_2} \le C \max_i \|A_i\|_{\psi_2}$. By definition of the sub-gaussian norm, $\left\|u^\top Av\right\|_{\psi_2} \le \|Av\|_{\psi_2}$. Since Av is a random vector with independent, mean-zero sub-gaussian coordinates A_iv, by Lemma 3.4.2 we have

$$\|Av\|_{\psi_2} \le \max_i \|A_iv\|_{\psi_2} \le \|A_i\|_{\psi_2}.$$

If the same assumptions are made about the columns of A instead of the rows, then we can simply consider $A^\top$ instead.

3.35 (Subgaussian norm of a sum) Proposition 2.7.1 extends to higher dimensions. Let $X_1, \dots, X_N$ be independent, zero-mean subgaussian vectors in $\mathbb{R}^n$. Show that

$$\left\|\sum_{i=1}^N X_i\right\|_{\psi_2}^2 \le C \sum_{i=1}^N \|X_i\|_{\psi_2}^2$$

where C is an absolute constant.

Solution: By Proposition 2.7.1, for any $u \in S^{n-1}$,

$$\left\|\left\langle u, \sum_{i=1}^N X_i \right\rangle\right\|_{\psi_2} = \left\|\sum_{i=1}^N \langle u, X_i\rangle\right\|_{\psi_2} \le C \sum_{i=1}^N \|\langle u, X_i\rangle\|_{\psi_2}.$$

Thus we have

$$\begin{aligned}
\left\|\sum_{i=1}^N X_i\right\|_{\psi_2} &= \sup_{u \in S^{n-1}} \left\|\left\langle u, \sum_{i=1}^N X_i \right\rangle\right\|_{\psi_2} \\
&\le C \sup_{u \in S^{n-1}} \sum_{i=1}^N \|\langle u, X_i\rangle\|_{\psi_2} \\
&\le C \sum_{i=1}^N \sup_{u \in S^{n-1}} \|\langle u, X_i\rangle\|_{\psi_2}
\end{aligned}$$

$$= C \sum_{i=1}^{N} \|X_i\|_{\psi_2} .$$

3.36 (Vectors with subgaussian coordinates) What happens if we drop the independence assumption in Lemma 3.4.2? X stays subgaussian, but the bound gets way worse:

(a) Let $X = (X_1, \dots, X_n)$ be a random vector with subgaussian (not necessarily independent) coordinates X_i. Show that X is subgaussian, and

$$\max_{i \le n} \|X_i\|_{\psi_2} \le \|X\|_{\psi_2} \le \sqrt{n} \max_{i \le n} \|X_i\|_{\psi_2}.$$

(b) Show by example that both bounds in part (a) are optimal in all dimensions n.

Solution: (a) The lower bound holds trivially. Now we prove the upper bound. For any $u \in S^{n-1}$,

$$\begin{aligned}
\|\langle X, u\rangle\|_{\psi_2}^2 &= \left\| \sum_{i=1}^{n} u_i X_i \right\|_{\psi_2}^2 \\
&\le \left(\sum_{i=1}^{n} \|u_i X_i\|_{\psi_2} \right)^2 \\
&\le \left(\sum_{i=1}^{n} u_i \|X_i\|_{\psi_2} \right)^2 \\
&\le \max_{i \le n} \|X_i\|_{\psi_2}^2 \left(\sum_{i=1}^{n} u_i \right)^2 \\
&\le n \|X_i\|_{\psi_2}^2 ,
\end{aligned}$$

where the last inequality is by Cauchy-Schwartz inequality. We finish the proof.

(b) Let $X \sim N(0, I_n)$, then for any $i \le n$, $X_i \sim N(0, 1)$. On the other hand, for any $u \in S^{n-1}$, $\langle u, X\rangle \sim N(0, 1)$, thus $\|X\|_{\psi_2} = \max_{i \le n} \|X_i\|_{\psi_2}$.

Let $Y = (y_0, y_0, \dots, y_0)$, where $y_0 > 0$ is a constant. Then $\|y_0\|_{\psi_2} = y_0/\sqrt{\log 2}$, but $\|Y\|_{\psi_2} = \sqrt{n} y_0/\sqrt{\log 2}$.

3.37 (The norm of subgaussian vectors needs not concentrate) It is tempting to conjecture that the norm concentration (Theorem 3.1.1) might apply more generally to any isotropic

subgaussian random vector X with $K = \|X\|_{\psi_2}$, so that coordinate independence would not be necessary. Show with an example that this is false.

Solution: Let $\theta \sim \text{Unif}(S^{n-1})$, and let ξ be an independent random variable such that $\mathbb{P}(\xi = \sqrt{2n}) = \mathbb{P}(\xi = 0) = 1/2$. Define the random vector $X = \xi\theta$. First, we check isotropy. Since $\mathbb{E}[\theta\theta^\top] = n^{-1}I_n$ and $\mathbb{E}[\xi^2] = n$, $\mathbb{E}[XX^\top] = \mathbb{E}[\xi^2]\mathbb{E}[\theta\theta^\top] = I_n$.

Second, we bound the sub-gaussian norm $K = \|X\|_{\psi_2}$. For any unit vector u, the projection is $\langle X, u\rangle = \xi\langle\theta, u\rangle$. Using $\|\langle\theta, u\rangle\|_{\psi_2} \le Cn^{-1/2}$,

$$\|\langle X, u\rangle\|_{\psi_2} \le \|\xi\|_\infty \|\langle\theta, u\rangle\|_{\psi_2} \le C\sqrt{2}.$$

Finally, consider the concentration of $\|X\|_2 = \xi$. As $|\xi - \sqrt{n}| \ge (\sqrt{2} - 1)\sqrt{n}$ almost surely,

$$\big\|\|X\|_2 - \sqrt{n}\big\|_{\psi_2} \gtrsim \sqrt{n} \gg K.$$

3.40 (Random vectors are almost orthogonal – sometimes.) In (3.14), we saw that two independent random vectors from the uniform distribution on the sphere S^{n-1} are likely almost orthogonal. Let's generalize this observation.

(a) Let X be a random vector in $\mathbb{R}^n$ with independent, zero-mean, unit-variance coordinates, all with subgaussian norms bounded by K. Define $\theta = X/\|X\|_2$ if $X \neq 0$, and $\theta = 0$ otherwise. Let η be an independent copy of θ. Prove that

$$|\langle\theta, \eta\rangle| \le \frac{C(K)}{\sqrt{n}}$$

with probability at least 0.99, where $C(K)$ may depend only on K.

(b) Show that the independence of coordinates is crucial in part (a). Find independent, identically distributed, isotropic random vectors X and Y in $\mathbb{R}^n$, whose subgaussian norms bounded by an absolute constant, and which satisfy $X = Y$ with probability at least 0.99.

Solution: (a) Set $S = \sum_{i=1}^n X_iY_i$. For each i, the product $Z_i = X_iY_i$ is mean-zero, unit-variance and is sub-exponential with $\|Z_i\|_{\psi_1} \le K^2$. Hence, by Bernstein's inequality, there exist constants $c_1 > 0$ such that for all $t > 0$,

$$\mathbb{P}(|S| \ge t) \le 2\exp\left(-c_1 \min\left\{\frac{t^2}{K^4 n}, \frac{t}{K^2}\right\}\right).$$

Choosing $t_0 = C_1 K^2 \sqrt{n}$ with C_1 sufficiently large gives

$$\mathbb{P}(|S| > t_0) \le 10^{-3}.$$

Moreover, by Theorem 3.1.1,

$$\big\| \|X\|_2 - \sqrt{n} \big\|_{\psi_2} \le CK^2, \qquad \big\| \|Y\|_2 - \sqrt{n} \big\|_{\psi_2} \le CK^2.$$

When $n \gg K^4$,

$$\mathbb{P}\left(|\|X\|_2 - \sqrt{n}| \le \tfrac{1}{4}\sqrt{n} \right) \le 10^{-3}, \qquad \mathbb{P}\left(|\|Y\|_2 - \sqrt{n}| \le \tfrac{1}{4}\sqrt{n} \right) \le 10^{-3}.$$

Combining the above results and using the union bound, with probability at least 0.99,

$$|\langle \theta, \eta \rangle| = \frac{|S|}{\|X\|_2 \|Y\|_2} \le \frac{C_2 K^2 \sqrt{n}}{n/16} = \frac{C(K)}{\sqrt{n}},$$

where $C(K) = 16 C_2 K^2$.

(b) Let $p = \sqrt{0.99}$ and let $g \sim N(0, I_n)$. Define

$$X = \begin{cases} 0, & \text{with probability } p, \\ \dfrac{g}{\sqrt{1-p}}, & \text{with probability } 1-p. \end{cases}$$

Let Y be an independent copy of X.

Then $\mathbb{E}X = 0$ and

$$\mathbb{E}[XX^\top] = (1-p) \cdot \mathbb{E}\left[\frac{gg^\top}{1-p} \right] = I_n,$$

so X is isotropic. Moreover,

$$\mathbb{P}(X = Y) \ge \mathbb{P}(X = 0,\ Y = 0) = p^2 = 0.99.$$

Conditioned on $X \neq 0$, each coordinate of X is Gaussian with variance $(1-p)^{-1}$; hence each coordinate has bounded ψ_2-norm by an absolute constant depending only on p.

3.41 (Exponentially many almost orthogonal vectors!) From linear algebra, we know there cannot be more than n orthogonal vectors in $\mathbb{R}^n$. But surprisingly, there are exponentially many

that are almost orthogonal! For any integer $N \le e^{cn}$, follow the steps below to find points $X_1, \dots, X_N \in S^{n-1}$ that satisfy

$$|\langle X_i, X_j \rangle| \le 0.01 \quad \text{for all distinct } i, j. \tag{3.39}$$

(a) Argue that independent random vectors $X, Y \sim \text{Unif}(S^{n-1})$ satisfy $|\langle X, Y \rangle| \le 0.01$ with probability at least $1 - 4\exp(-c_0 n)$, where c_0 is an absolute constant.

(b) Take a union bound over all pairs (i, j), $i \ne j$, and conclude that the event in (3.39) holds with positive probability.

Solution: (a) We write $X = u/\|u\|_2$ and $Y = v/\|v\|_2$, where u, v are independent standard Gaussians in $\mathbb{R}^n$. Write

$$|\langle X, Y \rangle| = \frac{|\langle u, v \rangle|}{\|u\|_2 \|v\|_2} = \frac{|\sum_{i=1}^n u_i v_i|}{\|u\|_2 \|v\|_2} \coloneqq \frac{|S|}{\|u\|_2 \|v\|_2}.$$

As for each i, u_i, v_i are independent standard Gaussians, all the $u_i v_i$ are independent, mean-zero, unit-variance, sub-exponential random variables and $\|u_i v_i\|_{\psi_1} = 8/3$. By Bernstein inequality,

$$\mathbb{P}(|S| \ge t) \le 2\exp\left(-c_1 \min\left\{\frac{9t^2}{64n}, \frac{3t}{8}\right\}\right).$$

Choosing $t_0 = 0.001n$ gives

$$\mathbb{P}(|S| > t_0) \le 2\exp\left(-\frac{c_1 \times 9 \times 10^{-6} n}{64}\right) \coloneqq \exp\left(-c_2 n\right).$$

Moreover, by Theorem 3.1.1,

$$\left\|\|u\|_2 - \sqrt{n}\right\|_{\psi_2} \le \frac{8C}{3}, \qquad \left\|\|v\|_2 - \sqrt{n}\right\|_{\psi_2} \le \frac{8C}{3}.$$

This implies

$$\mathbb{P}\left(\|u_2\| \le \sqrt{n}/2\right) \le \exp(-3c_3 n/256C^2) \coloneqq \exp(-c_4 n),$$
$$\mathbb{P}\left(\|v_2\| \le \sqrt{n}/2\right) \le \exp(-3c_3 n/256C^2) \coloneqq \exp(-c_4 n).$$

Combining the results and using the union bound, with probability at least $1 - \exp(-c_2 n) -$

$2\exp(-c_4 n)$,

$$|\langle X, Y\rangle| = \frac{|S|}{\|u\|_2 \|v\|_2} \le \frac{0.001n}{\frac{\sqrt{n}}{2} \cdot \frac{\sqrt{n}}{2}} \le 0.01.$$

(b) Set $\exp(-c_5 n) \ge \exp(-c_2 n) + 2\exp(-c_4 n)$. Take a union bound over all pairs (i, j), $i \ne j$,

$$\mathbb{P}\left(|\langle X_i, X_j\rangle| \le 0.01, \text{ for all } i \ne j\right) \ge 1 - \frac{N(N-1)}{2}\exp(-c_5 n).$$

Suppose $N \le e^{cn}$ and $c \le c_5/4$, then we can conclude $\mathbb{P}\left(|\langle X_i, X_j\rangle| \le 0.01, \text{ for all } i \ne j\right) > 0$.

3.43 (Subgaussian norm of the coordinate distribution) Let X be a random vector uniformly distributed on the standard vector basis of $\mathbb{R}^n$ for $n \ge 2$. Prove that

$$\|X\|_{\psi_2} \asymp \frac{1}{\sqrt{\log n}}$$

As always, "$\asymp$" means equivalence up to absolute constant factors.

Solution: For any $v \in S^{n-1}$,

$$\|\langle X, v\rangle\|_1 = \mathbb{E}\,|\langle X, v\rangle| = \frac{1}{n}\|v\|_1 \le \frac{1}{\sqrt{n}}, \quad \|\langle X, v\rangle\|_\infty = \operatorname{ess\,sup}|\langle X, v\rangle| = \|v\|_\infty \le 1.$$

By Exercise 2.35,

$$\|X\|_{\psi_2} \lesssim \frac{\|X\|_\infty}{\sqrt{\log(2\|X\|_\infty / \|X\|_1)}} \asymp \frac{1}{\sqrt{\log n}}.$$

On the other hand, $\langle X, e_1\rangle = 1$ with probability $1/n$ and $\langle X, e_1\rangle = 0$ with probability $1 - 1/n$. Therefore,

$$\|X\|_{\psi_2} \ge \|\langle X, e_1\rangle\|_{\psi_2} \asymp \frac{1}{\sqrt{\log n}},$$

where we use part (e) of Exercise 2.24. We finish the proof.

3.44 (The ℓ^1 ball is poorly subgaussian) Consider a ball in the ℓ^1 norm in $\mathbb{R}^n$:

$$K := \{x \in \mathbb{R}^n : \|x\|_1 \le r\}.$$

Let X be a random vector uniformly distributed in K.

(a) Compute the density of X_1, the first coordinate of X.

(b) Show that the random vector X is isotropic for some $r \asymp n$.

(c) For this value of r, show that $\|X\|_{\psi_2} > c\sqrt{n}$ where $c > 0$ is an absolute constant.

Solution: (a) Recall that the volume of a n-dimensional, radius-r ℓ^1 ball is $(2r)^n/n!$. Therefore, for $|u| \le r$,

$$\mathbb{P}(X_1 \le u) = \frac{n!}{(2r)^n} \int_{-r}^{u} \frac{(2(r-|x|))^{n-1}}{(n-1)!} dx = \begin{cases} \left(1+\frac{u}{r}\right)^n & -r \le u \le 0 \\ 1 - \left(1-\frac{u}{r}\right)^n & 0 \le u \le r \end{cases}.$$

Taking derivatives gives

$$p_{X_1}(u) = \frac{d\mathbb{P}(X_1 \le u)}{du} = \frac{n}{r}\left(1 - \frac{|u|}{r}\right)^{n-1}.$$

(b) By symmetry, $(X_1, X_2, \ldots, X_n)$ is identically distributed as $(-X_1, X_2, \ldots, X_n)$, thus $\mathbb{E}X_1X_j = 0$, $i = 2, \ldots, n$. Similarly, we can deduce that $\mathbb{E}X_iX_j = 0$, $i \ne j$. Using results in part (a),

$$\mathbb{E}X_1^2 = \frac{4r^2}{(n+2)(n+1)}.$$

Setting $r = \sqrt{(n+1)(n+2)}/2 \asymp n$ yields $\mathbb{E}X_1^2 = 1$. In this circumstance, we also have $\mathbb{E}X_i^2 = 1$, $i = 2, \ldots, n$ by symmetry.

(c) When $r = \sqrt{(n+1)(n+2)}/2 \asymp n$,

$$\mathbb{P}(X_1 \ge r/2) = \frac{1}{2^n} = e^{-c_1 n} \ge e^{-c_2 r^2/n},$$

where c_1, c_2 are absolute constants. We can conclude that $\|X\|_{\psi_2} \ge \|X_1\|_{\psi_2} \gtrsim \sqrt{n}$.

3.46 (Isotropic subgaussian distributions have high entropy) The entropy of a discrete random vector X that takes values x_i with probabilities p_i is defined as

$$H(X) = -\sum_i p_i \ln p_i.$$

Heuristically, $H(X)$ is proportional to the expected number of bits needed to encode X.

(a) (**Very discrete distributions have low entropy**) For any random X that takes N values, prove that $H(X) \le \ln N$.

(b) (**Isotropic subgaussian distributions have high entropy**) For any discrete, isotropic, subgaussian random vector X in $\mathbb{R}^n$, prove that

$$H(X) \geq \frac{n}{K^2} - \ln 2, \quad \text{where} \quad K = \|X\|_{\psi_2}.$$

(c) (**Isotropic subgaussian distributions have huge support**) Conclude that any random vector as in (b) must take at least $\frac{1}{2}\exp(K^{-2}n)$ different values.

Solution: (a) Construct a random variable Y such that $Y = p_i$ with probability $1/p_i$. By Jensen's inequality,

$$H(X) = \mathbb{E}(\log Y) \leq \log(\mathbb{E}Y) = \log N.$$

(b) Suppose $\mathbb{P}(X = x_i) = p_i$, then by Exercise 3.45,

$$p_i \leq 2\exp\left(-\frac{\|x_i\|^2}{K^2}\right).$$

Therefore,

$$\begin{aligned} H(x) &= \sum_i -p_i \log p_i \\ &\geq \sum_i p_i \left(\frac{\|x_i\|^2}{K^2} - \log 2\right) \\ &= \mathbb{E}\left[\frac{\|X\|_2^2}{K^2}\right] - \log 2 \\ &= \frac{n}{K^2} - \log 2. \end{aligned}$$

The last equality holds by isotropy.

(c) It must holds that

$$\frac{n}{K^2} - \log 2 \leq H(X) \leq \log N.$$

Thus $N \geq \frac{1}{2}\exp(K^{-2}n)$.

3.49 (Quadratic Grothendieck for PSD matrices) In this and next exercise, we relax the assumption of Grothendieck inequality by assuming $x_i = y_i$, as announced in Remark 3.5.3.

(a) (**Polarization identity**) Show that for any $n \times n$ symmetric matrix A and vectors $x, y \in \mathbb{R}^n$,

we have

$$x^\top Ay = \bar{x}^\top A\bar{x} - \bar{y}^\top A\bar{y} \quad \text{where} \quad \bar{x} = \frac{x+y}{2}, \quad \bar{y} = \frac{x-y}{2}.$$

(b) (**Quadratic Grothendieck**) Let $A = (a_{ij})$ be an $n \times n$ symmetric positive-semidefinite matrix. Assume that $\sum_{i,j} a_{ij}x_ix_j \leq 1$ for any numbers $x_i \in \{-1, 1\}$. Prove that for any Hilbert space H, we have $|\sum_{i,j} a_{ij}\langle u_i, v_j\rangle| \leq 2K$ for any unit vectors $u_i, v_j \in H$, where K is the absolute constant from Grothendieck inequality.

(c) Show the result in part (b) may fail without the positive-semidefinite assumption.

Solution: (a) Trivial.

(b) Since A is positive semidefinite, the function $f(z) = z^\top Az$ is convex. By Bauer's Maximum Principle, the maximum of a convex function over the compact convex set $[-1, 1]^n$ is attained at an extreme point. Thus:

$$\sup_{z\in[-1,1]^n} z^\top Az = \sup_{z\in\{-1,1\}^n} z^\top Az \leq 1.$$

Now consider any $x, y \in \{-1, 1\}^n$. Let $\bar{x} = (x + y)/2$ and $\bar{y} = (x - y)/2$. Note that $\bar{x}, \bar{y} \in [-1, 1]^n$. Using the polarization identity from part (a):

$$x^\top Ay = \bar{x}^\top A\bar{x} - \bar{y}^\top A\bar{y}.$$

Because A is positive semidefinite, $\bar{y}^\top A\bar{y} \geq 0$. Therefore:

$$x^\top Ay \leq \bar{x}^\top A\bar{x} \leq \sup_{z\in[-1,1]^n} z^\top Az \leq 1.$$

Finally, applying the Grothendieck inequality yields the conclusion.

(c) Let A be

$$\begin{bmatrix} 10K & 0 \\ 0 & -10K \end{bmatrix}$$

Then for any $x \in \{-1, 1\}^2$, $x^\top Ax = 0$. However, we simply take $u_1 = v_1 = (1, 0)^\top$, $u_2 = (0, 1)^\top$, $v_2 = (0, -1)^\top$, then $\sum_{i,j} a_{ij}\langle u_i, v_j\rangle = 20K$.

3.50 (Quadratic Grothendieck for diagonal-free matrices)

(a) (**Separate convexity**) A function $f(x_1, \ldots, x_n)$ is called *separately convex* if it is convex in each variable x_i. (Such function might not be convex – example?) Show that a separately

convex function $f : [-1,1]^n \to \mathbb{R}$ attains its maximum at some extremal point of the cube, i.e. at a point in $\{-1,1\}^n$.

(b) (**Quadratic Grothendieck**) Let $A = (a_{ij})$ be an $n \times n$ diagonal-free matrix. Assume that $|\sum_{i,j} a_{ij} x_i x_j| \leq 1$ for any numbers $x_i \in \{-1,1\}$. Prove that for any Hilbert space H, we have $|\sum_{i,j} a_{ij} \langle u_i, v_j \rangle| \leq 2K$ for any unit vectors $u_i, v_j \in H$, where K is the absolute constant from Grothendieck inequality.

Solution: (a) Suppose f attains its maximum at some point x^* such that $x_1^* \in (-1,1)$. By separate convexity,

$$\begin{aligned} f(x_1^*, x_2^*, \ldots, x_n^*) &= f\left(\frac{(x_1^*+1)}{2} - \left(1 - \frac{x_1^*+1}{2}\right), x_2^*, \ldots, x_n^*\right) \\ &\leq \frac{x_1^*+1}{2} f(1, x_2^*, \ldots, x_n^*) + \left(1 - \frac{x_1^*+1}{2}\right) f(-1, x_2^*, \ldots, x_n^*). \end{aligned}$$

Therefore, we have

$$f(1, x_2^*, \ldots, x_n^*) \vee f(-1, x_2^*, \ldots, x_n^*) \geq f(x_1^*, x_2^*, \ldots x_n^*).$$

We can repeat the procedure above for the remaining $(n-1)$ coordinates and conclude that there exists an extremal point $\tilde{x} \in \{-1,1\}^n$ such that $f(\tilde{x}) \geq f(x^*)$.

(b) It can be verified that the function $f(x) = x^\top A x$ is separately convex. Using the results in part (a), we have $\left|\sum_{i,j} a_{ij} x_i x_j\right| \leq 1$ for any $x \in [-1,1]^n$. Also, as the function $f(x) = -x^\top A x$ is separately convex, we have $\left|\sum_{i,j} a_{ij} x_i x_j\right| \geq -1$ for any $x \in [-1,1]^n$. Now consider any $x, y \in \{-1,1\}^n$. Let $\bar{x} = (x+y)/2$ and $\bar{y} = (x-y)/2$. Note that $\bar{x}, \bar{y} \in [-1,1]^n$. Using the polarization identity from part (a):

$$\left|x^\top A y\right| = \left|\bar{x}^\top A \bar{x} - \bar{y}^\top A \bar{y}\right| \leq \left|\bar{x}^\top A \bar{x}\right| + \left|\bar{y}^\top A \bar{y}\right| \leq 2.$$

The conclusion follows as a direct consequence of Grothendieck's inequality.

3.52 (SDP relaxation of a bilinear integer program) Consider the following integer optimization problem:

$$\text{maximize} \sum_{i,j=1}^{n} A_{ij} x_i y_j : \quad x_i, y_j = \pm 1 \text{ for } i, j \tag{3.40}$$

where A is an $n \times n$ matrix. Just like in (3.29), let us relax it to the following problem:

$$\text{maximize} \sum_{i,j} A_{ij} \langle X_i, Y_j \rangle : \quad \|X_i\|_2 = \|Y_j\|_2 = 1 \text{ for all } i, j, \tag{3.41}$$

where the maximum is over all unit vectors X_i, Y_j in $\mathbb{R}^k$ with some fixed dimension $k \in \mathbb{N}$.

(a) Express (3.41) as a semidefinite program.

(b) Show that the relaxation approximates the original problem (3.40) within an absolute constant factor.

Solution: (a) Write $X = (X_1, \ldots, X_n)$, $Y = (Y_1, \ldots, Y_n)$. Define

$$\tilde{A} = \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix},$$

and $Z = (X, Y)^T$. Then the (3.41) can be reformulated as

$$\max \operatorname{tr}(\tilde{A} Z Z^\top) : \quad (ZZ^\top)_{ii} = 1 \text{ for all i.}$$

The matrix $ZZ^\top$ is symmetric and positive-semidefinite. Since any symmetric and positive-semidefinite matrix admits a cholesky decomposition, (3.41) is equivalent with the following SDP problem:

$$\max_U \operatorname{tr}(\tilde{A} U) : \quad U \succeq 0, \ (U)_{ii} = 1 \text{ for all i.}$$

(b) The conclusion is a direct consequence of the Grothendieck inequality.

3.53 (Grothendieck identity) Prove Grothendieck identity (**Lemma 3.6.5**).

Solution: Let $\langle u, v \rangle = c$. Define $g' \sim N(0, I_2)$ $u' = (1, 0)^\top$, $v' = (c, \sqrt{1 - c^2})^\top$. We have

$$\begin{bmatrix} \langle g', u' \rangle \\ \langle g', v' \rangle \end{bmatrix} \overset{d}{=} \begin{bmatrix} \langle g, u \rangle \\ \langle g, v \rangle \end{bmatrix}.$$

It suffices to show $\mathbb{E} \operatorname{sign}(\langle g', u' \rangle) \operatorname{sign}(\langle g', v' \rangle) = \arcsin c$. Let $\theta = \arccos c$ denote the angle between u' and v', where $\theta \in [0, \pi]$. By rotational invariance of g', $\mathbb{P}(\operatorname{sign}(\langle g', u' \rangle) = \operatorname{sign}(\langle g', v' \rangle)) = (\pi - \theta)/\pi$,

and $\mathbb{P}(\mathsf{sign}(\langle g', u'\rangle) \neq \mathsf{sign}(\langle g', v'\rangle)) = \theta/\pi$. Therefore,

$$\mathbb{E}\,\mathsf{sign}(\langle g', u'\rangle)\,\mathsf{sign}(\langle g', v'\rangle) = \frac{\pi - 2\theta}{\pi} = \frac{2}{\pi}\left(\frac{\pi}{2} - \theta\right) = \frac{2}{\pi}\arcsin c.$$

3.54 (Approximating a max cut: deterministic outcome, random runtime) Goemans-Williamson algorithm (Theorem 3.6.4) finds a cut that is large only *in expectation*. For any $\varepsilon > 0$, give an $(0.878 - \varepsilon)$-approximation algorithm for the maximum cut that *always* finds a valid cut but may have random running time. Bound the expected running time.

Solution: We cut G repeatedly with the Geomans-Williamson algorithm until $\operatorname{cut}(G, x) \geq (0.878 - \varepsilon)\operatorname{sdp}(G)$. We define a "success" as finding a cut satisfying the condition above. For each cut, we establish the lower bound of the probability of success :

$$\begin{aligned}
\mathbb{P}(\operatorname{cut}(G, x) \geq (0.878 - \varepsilon)\operatorname{sdp}(G)) &\geq \mathbb{P}\left(\operatorname{cut}(G, x) \geq \frac{(0.878 - \varepsilon)}{0.878}\mathbb{E}\operatorname{cut}(G, x)\right) \\
&\geq \left(1 - \frac{0.878 - \varepsilon}{0.878}\right)^2 \frac{(\mathbb{E}\operatorname{cut}(G, x))^2}{\mathbb{E}\,(\operatorname{cut}(G, x))^2} \\
&\geq \left(1 - \frac{0.878 - \varepsilon}{0.878}\right)^2 \frac{(0.878\operatorname{sdp}(G, x))^2}{(\operatorname{sdp}(G, x))^2} \\
&\geq \varepsilon^2,
\end{aligned}$$

where the first inequality holds by the fact $\mathbb{E}\operatorname{cut}(G, x) \geq 0.878\operatorname{sdp}(G)$, the second inequality holds by the Paley-Zygmund inequality and the third inequality is due to $\mathbb{E}\operatorname{cut}(G, x) \geq 0.878\operatorname{sdp}(G)$ and $\operatorname{cut}(G, x) \leq \operatorname{sdp}(G)$. Let T be the number of iterations required to achieve success. T follows a Geometric distribution with parameter $p \geq \varepsilon^2$. Thus, the expected number of iterations is bounded by:

$$\mathbb{E}[T] = \frac{1}{p} \leq \frac{1}{\varepsilon^2}.$$

3.56 (Linearizing Grothendieck identity) Let's get rid of the nonlinearity "arcsin" in Grothendieck identity (Lemma 3.6.5) by adding a correction term. Prove the following for a random vector $g \sim \mathcal{N}(0, I_n)$ and any fixed unit vectors $u, v \in \mathbb{R}^{n-1}$:

(a) $\mathbb{E}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle = \sqrt{\frac{2}{\pi}}\langle u, v\rangle$.

(b) $\mathbb{E}\operatorname{sign}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle = \frac{2}{\pi}\langle u, v\rangle + \mathbb{E}[Z_u Z_v]$, where $Z_w = \sqrt{\frac{2}{\pi}}\langle g, w\rangle - \operatorname{sign}\langle g, w\rangle$.

Solution: (a) Following arguments similar to those in the solution of Exercise 3.54, without

loss of generality we may assume that $g \sim N(0, I_2)$, $v = (1, 0)$, and $u = (u_1, u_2)$. We have

$$\begin{aligned}\mathbb{E}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle &= \mathbb{E}\operatorname{sign}(g_1)(g_1u_1 + g_2u_2)\\ &= \mathbb{E}\operatorname{sign}(g_1)g_1u_1 + \mathbb{E}\operatorname{sign}(g_1)g_2u_2.\end{aligned}$$

Since $g \sim N(0, I_2)$, $\mathbb{E}\operatorname{sign}(g_1)g_2u_2 = 0$. Thus

$$\begin{aligned}\mathbb{E}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle &= \mathbb{E}\operatorname{sign}(g_1)g_1u_1\\ &= u_1\mathbb{E}|g_1|\\ &= \langle u, v\rangle\sqrt{\frac{2}{\pi}}.\end{aligned}$$

(b) We have

$$\begin{aligned}\mathbb{E}[Z_uZ_v] =& \frac{2}{\pi}\mathbb{E}[\langle g, u\rangle\langle g, v\rangle] + \mathbb{E}[\operatorname{sign}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle]\\ &- \sqrt{\frac{2}{\pi}}\mathbb{E}[\langle g, u\rangle \operatorname{sign}\langle g, v\rangle] - \sqrt{\frac{2}{\pi}}\mathbb{E}[\langle g, v\rangle \operatorname{sign}\langle g, u\rangle]\\ =& \frac{2}{\pi}\langle u, v\rangle + \mathbb{E}[\operatorname{sign}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle] - \frac{4}{\pi}\langle u, v\rangle\\ =& \mathbb{E}[\operatorname{sign}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle] - \frac{2}{\pi}\langle u, v\rangle.\end{aligned}$$

Here we use $\mathbb{E}[\langle g, u\rangle\langle g, v\rangle] = u^\top\mathbb{E}[gg^\top]v = \langle u, v\rangle$, and $\mathbb{E}[\langle g, u\rangle \operatorname{sign}\langle g, v\rangle] = \sqrt{2/\pi}\langle u, v\rangle$ shown in part (a). We finish the proof.

3.57 (A sharper PSD Grothendieck) For positive semidefinite matrices, Grothendieck inequality (Theorem 3.5.1) holds with a better constant $\pi/2 \approx 1.571$, with a simpler proof. Let's discover it. Let $A = (a_{ij})$ be an $n \times n$ symmetric positive-semidefinite matrix. Suppose $\sum_{i,j} a_{ij}x_ix_j \le 1$ for any numbers $x_i \in \{-1, 1\}$. Use Exercise 3.56 to show that for any Hilbert space H, we have $\sum_{i,j} a_{ij}\langle u_i, u_j\rangle \le \pi/2$ for any unit vectors $u_i \in H$.

Solution: By Exercise 3.56,

$$\begin{aligned}\sum_{i,j} a_{ij}\langle u_i, u_j\rangle =& \sum_{i,j} \frac{\pi}{2} \cdot a_{ij}\left(\mathbb{E}[\operatorname{sign}\langle g, u_i\rangle \operatorname{sign}\langle g, u_j\rangle] - \mathbb{E}[Z_{u_i}Z_{u_j}]\right)\\ =& \frac{\pi}{2}\,\mathbb{E}\left[\sum_{i,j} a_{ij}\left([\operatorname{sign}\langle g, u_i\rangle \operatorname{sign}\langle g, u_j\rangle]\right)\right] - \frac{\pi}{2}\,\mathbb{E}\left[\sum_{i,j} a_{ij}[Z_{u_i}Z_{u_j}]\right].\end{aligned}$$

Since A is positive-semidefinite, we always have $\sum_{i,j} a_{ij}[Z_{u_i} Z_{u_j}] \geq 0$. By assumption of the theorem, it always holds that $\sum_{i,j} a_{ij} ([\text{sign}\langle g, u_i\rangle \, \text{sign}\langle g, u_j\rangle]) \leq 1$. Therefore, we have $\sum_{i,j} a_{ij} \langle u_i, u_j\rangle \leq \pi/2$.

4 Random Matrices

4.2 (Basic properties of the operator norm)

(a) Prove that the operator norm $\|A\|$ introduced in Definition 4.1.8 is indeed a norm on the space of $m \times n$ matrices.

(b) For any matrix, show that

$$\|A^\top\| = \|A\|.$$

(c) For any two matrices of appropriate dimensions, show that

$$\|AB\| \leq \|A\| \cdot \|B\|.$$

Find an example where the left side is zero, but the right side is not.

(d) Prove that the operator norm of any submatrix is bounded by the operator norm of the matrix.

Solution: (a) The definition in Definition 4.1.8 is equivalent to $\|A\| \coloneqq \sup_{\|x\|_2=1} \|Ax\|_2$. By definition, $\|A\|$ is non-negative. If $\|A\| = 0$, then $\|Ax\|_2 = 0$ for all x, implying $Ax = 0$ for all x, so $A = 0$. Also, for any scalar α,

$$\|\alpha A\| = \sup_{\|x\|_2=1} \|\alpha A x\|_2 = |\alpha| \sup_{\|x\|_2=1} \|Ax\|_2 = |\alpha| \|A\|.$$

Finally, for any unit vector x,

$$\|(A+B)x\|_2 \leq \|Ax\|_2 + \|Bx\|_2 \leq \|A\| + \|B\|.$$

Taking the supremum over all unit vectors yields $\|A + B\| \leq \|A\| + \|B\|$.

(b) Using the variational characterization of the Euclidean norm:

$$\|A\| = \sup_{\|x\|_2=1} \|Ax\|_2 = \sup_{\|x\|_2=1} \sup_{\|y\|_2=1} \langle Ax, y\rangle.$$

Observe that $\langle Ax, y\rangle = \langle x, A^\top y\rangle$. Therefore:

$$\|A\| = \sup_{\|y\|_2=1} \sup_{\|x\|_2=1} \langle x, A^\top y\rangle = \sup_{\|y\|_2=1} \|A^\top y\|_2 = \|A^\top\|.$$

(c) For any x, $\|(AB)x\|_2 \le \|A\|\|Bx\|_2 \le \|A\|\|B\|\|x\|_2$. Thus $\|AB\| \le \|A\|\|B\|$.

Counterexample: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have $\|A\| = 1$ and $\|B\| = 1$, but $AB = 0$.

(d) Let A' be a submatrix of A. Let y be a unit vector maximizing the norm on the subspace of the submatrix, i.e., $\|A'y\|_2 = \|A'\|$. Define x by zero-padding y to match the dimensions of A. Then $\|x\|_2 = \|y\|_2 = 1$. Since Ax contains all entries of $A'y$ (and possibly more), $\|Ax\|_2 \ge \|A'y\|_2$. Therefore:

$$\|A\| \ge \|Ax\|_2 \ge \|A'y\|_2 = \|A'\|.$$

4.4 (Operator norm vs. Frobenius norms) Let A be an $m \times n$ matrix.

(a) If A has rank r, prove that

$$\|A\| \le \|A\|_F \le \sqrt{r}\|A\|.$$

Show that both bounds are achievable for any m, n and $1 \le r \le \min(m, n)$.

(b) If Z is an isotropic random vector in $\mathbb{R}^n$, show that

$$\mathbb{E}\|AZ\|_2^2 = \|A\|_F^2.$$

(c) If B is a $k \times m$ matrix, prove that

$$\|BA\|_F \le \|B\|\|A\|_F.$$

(d) If A is a diagonal matrix, improve (c) to

$$\|BA\|_F \le \|B\|_{1\to 2}\|A\|_F,$$

where the $\|B\|_{1\to 2}$ denotes the maximal Euclidean norm of columns of B.

Solution: (a) Let $s_1 \geq s_2 \geq \cdots \geq s_r > 0$ be r non-zero singular values of A, then $\|A\| = s_1$ and $\|A\|_F = \sqrt{\sum_{i=1}^r s_i^2}$. It is straightforward to see $\|A\| \leq \|A\|_F \leq \sqrt{r}\,\|A\|$. We construct

$$A_1 = \begin{bmatrix} I_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0_{1\times(n-1)} \\ 0_{(m-1)\times 1} & 0_{(m-1)\times(n-1)} \end{bmatrix}$$

then $\|A_2\|_F = \|A_2\|$ and $\|A_1\|_F = \sqrt{r}\,\|A_1\|$.

(b) We have

$$\begin{aligned} \mathbb{E}\,\|AZ\|_2^2 &= \mathbb{E}(AZ)^\top(AZ) \\ &= \mathbb{E}Z^\top A^\top AZ \\ &= \mathbb{E}\,\mathrm{tr}(Z^\top A^\top AZ) \\ &= \mathbb{E}\,\mathrm{tr}(A^\top AZZ^\top) \\ &= \mathrm{tr}(A^\top A\mathbb{E}ZZ^\top) \\ &= \mathrm{tr}(A^\top A) \\ &= \|\mathrm{vec}(A)\|_2^2 \\ &= \|A\|_F^2 . \end{aligned}$$

The fifth equality holds due to the linearity of expectation.

(c) Let $g \sim N(0, I_n)$, by part (b), we have

$$\begin{aligned} \|BA\|_F^2 &= \mathbb{E}\,\|BAg\|_2^2 \\ &\leq \mathbb{E}\,\|B\|^2\,\|Ag\|_2^2 \\ &= \|B\|^2\,\|A\|_F^2 \\ &\leq \|B\|_F^2\,\|A\|_F^2 . \end{aligned}$$

(d) Let $A = \mathrm{diag}\{a_1, \ldots, a_m\}$ and $B = (b_{\cdot 1}, b_{\cdot 2}, \ldots, b_{\cdot m})$, then $BA = (a_1 b_{\cdot 1}, a_2 b_{\cdot 2}, \ldots, a_m b_{\cdot m})$. We can get

$$\|BA\|_F^2 = \sum_{i=1}^m a_i^2\,\|b_{\cdot i}\|_2^2 \leq \|B\|_{1\to 2}^2 \sum_{i=1}^m a_i^2 = \|B\|_{1\to 2}^2\,\|A\|_F^2 .$$

4.6 (Power method) Here is a computationally friendly way to approximate the operator norm of a matrix without computing the spectrum. Consider an $n \times n$ matrix A and a random vector $x \sim N(0, I_n)$. Show that, with probability 1, we have

$$\sqrt[k]{\|A^k x\|_2} \to \|A\| \quad \text{as } k \to \infty.$$

Solution: The original problem statement is false. Consider the counterexample

$$A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}.$$

Instead, we assume A is symmetric, Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A, and let $v_1, \ldots, v_n$ be the corresponding orthonormal eigenvectors. The random vector x can be expanded as $x = \sum_{i=1}^n c_i v_i$, where $c_i = \langle x, v_i \rangle$. Since x is standard Gaussian, $c_1 \neq 0$ with probability 1. Applying A^k to the vector x, we obtain $A^k x = \sum_{i=1}^n c_i \lambda_i^k v_i$. Therefore,

$$\sqrt[k]{\|A^k x\|_2} = |\lambda_1| \left(c_1^2 + \sum_{i=2}^n c_i^2 \left(\frac{\lambda_i}{\lambda_1} \right)^{2k} \right)^{\frac{1}{2k}}.$$

Let $R_k = c_1^2 + \sum_{i=2}^n c_i^2 (\lambda_i/\lambda_1)^{2k}$. Observe that $|\lambda_i/\lambda_1| \leq 1$ for all i and $c_1 \neq 0$ almost surely, $\lim_{k\to\infty} R_k^{1/2k} = 1$. Therefore,

$$\lim_{k\to\infty} \sqrt[k]{\|A^k x\|_2} = |\lambda_1|.$$

Since $\|A\| = |\lambda_1|$, the result follows.

4.7 (Operator norm vs. the norms of columns) The operator norm of a matrix is hard to express in terms of the entries, but in this and next exercises, you will prove some useful bounds. Let A be an $m \times n$ matrix with columns $A_1, \ldots, A_n$.

(a) Show that

$$\|A\| \geq \max_i \|A_i\|_2.$$

(b) Prove that if $\|A\| = \|A_i\|_2$ for some i, then A_i is orthogonal to all other columns.

(c) Argue that the same is true for the rows.

Solution:(a) The conclusion follows by noting that $A_i = Ae_i$, where e_i is the i-th standard

basis in $\mathbb{R}^n$.

(b) WLOG, we assume $i \neq 1$. Suppose A_1 is not orthogonal to A_2. Let $x_\varepsilon = (1, \varepsilon, 0, \ldots, 0)$,

$$\begin{aligned}\|Ax_\varepsilon\|_2^2 - \|A\|_2^2 \|x_\varepsilon\|_2^2 &= \langle A_1 + \varepsilon A_2, A_1 + \varepsilon A_2\rangle - \|A_1\|_2^2 (1+\varepsilon^2) \\ &= 2\varepsilon\langle A_1, A_2\rangle + \varepsilon^2(\|A_2\|_2^2 - \|A_1\|_2^2).\end{aligned}$$

Setting $\varepsilon = \langle A_1, A_2\rangle/(\|A_1\|_2^2 - \|A_2\|_2^2)$, we have $\|Ax_\varepsilon\|_2^2 - \|A\|_2^2 \|x_\varepsilon\|_2^2 < 0$, which contradicts the definition of $\|A\|$.

(c) Since $\|A\| = \left\|A^\top\right\|$, we can use the results in part (a) and part (b) for $A^\top$ to draw the conclusion.

4.8 (Schur bound) For an $m \times n$ matrix A with rows $A_{i:}$ and columns $A_{:j}$, prove that

$$\|A\| \leq \sqrt{\max_i \|A_{i:}\|_1 \cdot \max_j \|A_{:j}\|_1}.$$

Solution: Recall that $\|A\| = \sup_{x\in S^m, y\in S^n} |x^\top A y|$. For any $x \in S^m$, $y \in S^n$, we have

$$\begin{aligned}|x^\top A y| &= \left|\sum_{i,j} x_i A_{ij} y_j\right| \\ &\leq \sum_{i,j} |x_i A_{ij} y_j| \\ &= \sum_{i,j} \left|x_i \sqrt{|A_{ij}|}\right| \cdot \left|y_j \sqrt{|A_{ij}|}\right| \\ &\leq \sqrt{\sum_{i,j} x_i^2 |A_{ij}|} \cdot \sqrt{\sum_{i,j} y_j^2 |A_{ij}|} \\ &\leq \sqrt{\sum_i x_i^2 \max_i \|A_{i:}\|_1} \cdot \sqrt{\sum_j y_j^2 \max_j \|A_{:j}\|_1} \\ &\leq \sqrt{\max_i \|A_{i:}\|_1 \max_j j \, \|A_{:j}\|_1},\end{aligned}$$

where the second inequality holds by Cauchy-Schwartz inequality, the third inequality holds by definition of the $\|\cdot\|_1$ norm, and the last inequality holds by noting that $x \in S^m$ and $y \in S^n$.

4.9 (Walsh matrices) Let's construct the "most spread" orthogonal matrices, whose all entries have the same absolute value. Start with the 2×2 matrix $W_1 := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and then iteratively define the block matrix

$$W_{k+1} := \begin{bmatrix} W_k & W_k \\ W_k & -W_k \end{bmatrix}, \quad k = 1, 2, \ldots$$

Thus W_2 is a 4×4 matrix, W_3 is an 8×8 matrix, and so on. These matrices, all with ± 1 entries, are called Walsh matrices. Prove that $\frac{1}{\sqrt{2^k}} W_k$ is an orthogonal matrix for each k.

Solution: We proceed by induction on k. First, it is straightforward to verify W_1 is orthogonal. Assuming $\frac{1}{\sqrt{2^k}} W_k$ is orthogonal, then

$$\begin{aligned} \left(\frac{1}{\sqrt{2^{k+1}}} W_{k+1}^\top \right) \left(\frac{1}{\sqrt{2^{k+1}}} W_{k+1} \right) &= \frac{1}{2^{k+1}} \begin{bmatrix} 2W_k^\top W_k & 0 \\ 0 & 2W_k^\top W_k \end{bmatrix} \\ &= \begin{bmatrix} I_{2^k} & 0 \\ 0 & I_{2^k} \end{bmatrix} \\ &= I_{2^{k+1}}. \end{aligned}$$

We finish the proof.

4.10 (Matrices with ± 1 entries) Prove that any $n \times n$ matrix A with ± 1 entries satisfies

$$\sqrt{n} \le \|A\| \le n,$$

and both bounds are achievable for infinitely many n.

Solution: The upper bound $\|A\| \le n$ holds by Schur bound (Exercise 4.8), and the lower bound $\|A\| \ge \sqrt{n}$ holds by part (a) in Exercise 4.7. To prove achievability, we first consider the matrix $\mathbf{1}_n \mathbf{1}_n^\top$, where we use $\mathbf{1}_n$ to denote the all-one vector in $\mathbb{R}^n$. Observe that $\left\| \mathbf{1}_n \mathbf{1}_n^\top (\frac{1}{\sqrt{n}} \mathbf{1}_n) \right\|_2 = \|\sqrt{n} \mathbf{1}_n\|_2 = n$, we have $\left\| \mathbf{1}_n \mathbf{1}_n^\top \right\| \ge n$. On the other hand, the Walsh matrix $W_k \in \mathbb{R}^{2^k}$ satisfies $\|W_k\| = \sqrt{2^k}$ by Exercise 4.9.

4.11 (Difference of orthogonal projections) Prove that any two orthogonal projections P and Q in $\mathbb{R}^n$ satisfy $\|P - Q\| \le 1$. And if you are up for a challenge, try the next exercise for a refined bound.

Solution: For any $x \in S^{n-1}$,

$$\begin{aligned}\|Px - x/2\|_2^2 &= \left(Px - \frac{x}{2}\right)^\top \left(Px - \frac{x}{2}\right) \\ &= x^\top P^\top Px + \frac{x^\top x}{4} - \frac{x^\top Px}{2} - \frac{x^\top P^\top x}{2} \\ &= (Px)^\top (Px - x) + \frac{x^\top x}{4} \\ &= \frac{1}{4}.\end{aligned}$$

The second last inequality holds since the projection residual $x - Px$ is orthogonal to the column space of P. Similarly, $\|Qx - x/2\|_2^2 = 1/4$. By the triangular inequality $\|(Q-P)x\|_2 \leq \|Qx - x/2\|_2 + \|Px - x/2\|_2 = 1$. The conclusion follows.

4.12 (Difference vs. products of orthogonal projections) For any two orthogonal projections P and Q in $\mathbb{R}^n$, prove the following.

(a) $\|P - Q\| = \max(\|P_\perp Q\|, \|PQ_\perp\|)$, where $P_\perp = I_n - P$ and $Q_\perp = I_n - Q$.

(b) If the ranks of P and Q are different, then $\|P - Q\| = 1$.

(c) If the ranks of P and Q are the same, then $\|P - Q\| = \|P_\perp Q\| = \|PQ_\perp\|$.

Solution: (a) Take an orthogonal matrix $\bar{U} = \begin{bmatrix} U & U_\perp \end{bmatrix}$ whose first columns form U and span $\operatorname{Im} P$. Similarly, take an orthogonal matrix $\bar{V} = \begin{bmatrix} V & V_\perp \end{bmatrix}$ whose first columns form V and span $\operatorname{Im} Q$.

Then we have $P = UU^\top$, $P_\perp = U_\perp U_\perp^\top$, $PU = U$, $PU_\perp = 0$. Similarly, $Q = VV^\top$, $Q_\perp = V_\perp V_\perp^\top$, $QV = V$, $QV_\perp = 0$. Since the operator norm is unitarily invariant, $\|P - Q\| = \|\bar{U}^\top (P - Q)\bar{V}\|$. We compute this product in block form:

$$\begin{aligned}\bar{U}^\top (P - Q)\bar{V} &= \begin{bmatrix} U^\top (P-Q)V & U^\top (P-Q)V_\perp \\ U_\perp^\top (P-Q)V & U_\perp^\top (P-Q)V_\perp \end{bmatrix} \\ &= \begin{bmatrix} 0 & U^\top V_\perp \\ -U_\perp^\top V & 0 \end{bmatrix}\end{aligned}$$

The norm of a block-antidiagonal matrix is the maximum of the norms of its blocks. Observing

that $\|U^\top V_\perp\| = \|UU^\top V_\perp\| = \|PQ_\perp\|$ and $\|U_\perp^\top V\| = \|U_\perp U_\perp^\top V\| = \|P_\perp Q\|$, we obtain

$$\|P - Q\| = \max(\|PQ_\perp\|, \|P_\perp Q\|).$$

(b) Let $C = U^\top V$. Then $\|PQ_\perp\|^2 = \|U^\top V_\perp\|^2 = \|U^\top (I - VV^\top)U\| = \|I - CC^\top\|$. If $\text{rank}(P) \neq \text{rank}(Q)$, the matrix C is not square. Assuming $\text{rank}(P) > \text{rank}(Q)$, $CC^\top$ is singular, so 0 is an eigenvalue of $CC^\top$. Consequently, 1 is an eigenvalue of $I - CC^\top$, implying $\|PQ_\perp\| = 1$. Thus $\|P - Q\| = 1$.

(c) If $\text{rank}(P) = \text{rank}(Q)$, then C is square. The non-zero eigenvalues of $CC^\top$ and $C^\top C$ are identical. Therefore, $\|PQ_\perp\|^2 = \|I - CC^\top\| = \|I - C^\top C\| = \|P_\perp Q\|^2$. It follows that $\|P - Q\| = \|P_\perp Q\| = \|PQ_\perp\|$.

4.13 (Davis-Kahan for spectral projections) Let us prove a version of Davis-Kahan theorem (Theorem 4.1.15) for projections on the top k eigenvectors. Consider two symmetric matrices A and B with spectral decompositions $A = \sum_i \lambda_i u_i u_i^\top$ and $B = \sum_i \mu_i v_i v_i^\top$, in which the eigenvalues are (weakly) decreasing. Prove that the spectral projections $P_A = \sum_{i=1}^k u_i u_i^\top$ and $P_B = \sum_{i=1}^k v_i v_i^\top$ satisfy

$$\|P_A - P_B\| \leq \frac{2\|A - B\|}{\lambda_k - \lambda_{k+1}}.$$

Solution: WLOG we assume $\lambda_k - \lambda_{k-1} = \delta > 0$. Since $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ are both orthogonal bases of $\mathbb{R}^n$, it follows that $\text{rank}(P_A) = \text{rank}(P_B)$. By Exercise 4.12, $\|P_A - P_B\| = \|P_A(I - P_B)\|$. Observe that $I - P_B = \sum_{j=k+1}^n v_i v_i^\top$. Let $I = [\lambda_k, \infty)$ and $J = (-\infty, \mu_{k+1}]$. We assume $\|A - B\| \leq \delta/2$, otherwise the conclusion will become trivial as it always holds that $\|P_A - P_B\| \leq 1$. Therefore, we have

$$\lambda_k - \mu_{k+1} \geq |\lambda_k - \lambda_{k+1}| - |\lambda_{k+1} - \mu_{k+1}| \geq \delta - \|A - B\| \geq \frac{\delta}{2},$$

where the second inequality holds by definition of δ and Weyl's eigenvalue perturbation inequality. Therefore, I and J are $\delta/2$-separated and we can directly use Lemma 4.1.16 to conclude.

$$\|P_A(I - P_B)\| \leq \frac{\|A - B\|}{\delta/2}.$$

The desired inequality follows immediately.

4.14 (Hermitian dilation) Two common ways to turn any $m \times n$ matrix A into a symmetric one are: (1) use $A^\top A$ or $AA^\top$, or (2) take its *Hermitian dilation*, defined as the $(m+n) \times (m+n)$ block matrix

$$H = \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix}.$$

Hermitian dilation has several advantages: it is a linear transformation and it preserves sparsity. Let $A = \sum_i s_i u_i v_i^\top$ be a singular value decomposition. Prove that the only nonzero eigenvalues of H are of the form $\pm s_i$, with corresponding eigenvectors $\left[\begin{smallmatrix} u_i \\ \pm v_i \end{smallmatrix}\right]$.

Solution: Let $\operatorname{rank}(A) = r$. It follows immediately that $\operatorname{rank}(H) = 2r$, and therefore H has at most $2r$ distinct nonzero eigenvalues. Now we show that the spectral decomposition of H is

$$H = \sum_{i=1}^{r} s_i \begin{bmatrix} u_i/\sqrt{2} \\ v_i/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_i/\sqrt{2} \\ v_i/\sqrt{2} \end{bmatrix}^\top + \sum_{i=1}^{r} (-s_i) \begin{bmatrix} u_i/\sqrt{2} \\ -v_i/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_i/\sqrt{2} \\ -v_i/\sqrt{2} \end{bmatrix}^\top$$

First, it is straightforward to verify $\left\{\left[\begin{smallmatrix} u_i \\ \pm v_i \end{smallmatrix}\right]\right\}_{i=1}^r$ forms an orthogonal system. Next, we have

$$\begin{aligned}
&\sum_{i=1}^{r} s_i \begin{bmatrix} u_i/\sqrt{2} \\ v_i/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_i/\sqrt{2} \\ v_i/\sqrt{2} \end{bmatrix}^\top + \sum_{i=1}^{r} (-s_i) \begin{bmatrix} u_i/\sqrt{2} \\ -v_i/\sqrt{2} \end{bmatrix} \begin{bmatrix} u_i/\sqrt{2} \\ -v_i/\sqrt{2} \end{bmatrix}^\top \\
&= \begin{bmatrix} \frac{1}{2}\sum_{i=1}^r s_i u_i u_i^\top + \frac{1}{2}\sum_{i=1}^r (-s_i) u_i u_i^\top & \frac{1}{2}\sum_{i=1}^r s_i u_i v_i^\top + \frac{1}{2}\sum_{i=1}^r (-s_i) u_i (-v_i)^\top \\ \frac{1}{2}\sum_{i=1}^r s_i v_i u_i^\top + \frac{1}{2}\sum_{i=1}^r (-s_i)(-v_i) u_i^\top & \frac{1}{2}\sum_{i=1}^r s_i v_i v_i^\top + \frac{1}{2}\sum_{i=1}^r (-s_i)(-v_i)(-v_i)^\top \end{bmatrix} \\
&= \begin{bmatrix} 0 & A \\ A^\top & 0 \end{bmatrix}.
\end{aligned}$$

We have finished the proof.

4.15 (Wedin theorem) Davis-Kahan inequality (Theorem 4.1.15) applies for symmetric matrices. Let's prove a version for general rectangular matrices. Let A and B be $m \times n$ matrices with singular value decompositions $A = \sum_i s_i u_i v_i^\top$ and $B = \sum_i t_i w_i z_i^\top$, where the singular values are (weakly) decreasing. Prove that the projections on the top k left singular vectors $P_A = \sum_{i=1}^k u_i u_i^\top$ and $P_B = \sum_{i=1}^k w_i w_i^\top$ satisfy

$$\|P_A - P_B\| \le \frac{2\|A - B\|}{s_k - s_{k+1}}.$$

Get a similar bound for the projections on the top k right singular vectors. Also, prove a version of Davis-Kahan inequality (Theorem 4.1.15) on the k-th singular vectors.

Solution: Let $H_A = \left[\begin{smallmatrix} 0 & A \\ A^\top & 0 \end{smallmatrix}\right]$ be the Hermitian dilation of A. Similarly, let $H_B = \left[\begin{smallmatrix} 0 & B \\ B^\top & 0 \end{smallmatrix}\right]$ be the Hermitian dilation of B. WLOG, we assume $\operatorname{rank}(A) = r > k$. According to Exercise 4.14, the top k eigenvalues of H_A are $s_1, \ldots, s_k$, with corresponding eigenvectors $\left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} u_k \\ v_k \end{smallmatrix}\right]$. And the top k eigenvalues of H_B are $t_1, \ldots, t_k$, with corresponding eigenvectors $\left[\begin{smallmatrix} w_1 \\ z_1 \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} w_k \\ z_k \end{smallmatrix}\right]$. Define $P_{HA} = \sum_{i=1}^k \left[\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right]\left[\begin{smallmatrix} u_i \\ v_i \end{smallmatrix}\right]^\top$ and $P_{HB} = \sum_{i=1}^k \left[\begin{smallmatrix} w_i \\ z_i \end{smallmatrix}\right]\left[\begin{smallmatrix} w_i \\ z_i \end{smallmatrix}\right]^\top$. According to Exercise 4.13, we have

$$\|P_{HA} - P_{HB}\| \leq \frac{2\,\|H_A - H_B\|}{s_k - s_{k-1}}.$$

Now we claim that $\|P_{HA} - P_{HB}\| \geq \|P_A - P_B\|$ and $\|H_A - H_B\| = \|A - B\|$. Indeed, we have

$$\begin{aligned}
\|P_{HA} - P_{HB}\| &= \left\| \begin{bmatrix} \sum_{i=1}^k u_i u_i^\top - \sum_{i=1}^k w_i w_i^\top & \sum_{i=1}^k u_i v_i^\top - \sum_{i=1}^k w_i z_i^\top \\ \sum_{i=1}^k v_i u_i^\top - \sum_{i=1}^k z_i w_i^\top & \sum_{i=1}^k v_i v_i^\top - \sum_{i=1}^k z_i z_i^\top \end{bmatrix} \right\| \\
&\geq \left\| \sum_{i=1}^k u_i u_i^\top - \sum_{i=1}^k w_i w_i^\top \right\| \\
&= \|P_A - P_B\|.
\end{aligned}$$

And

$$\|H_A - H_B\| = \left\| \begin{bmatrix} 0 & A - B \\ (A-B)^\top & 0 \end{bmatrix} \right\| = \|A - B\|$$

since $H_A - H_B$ is block-wise anti-diagonal and $\|A - B\| = \left\|(A-B)^\top\right\|$.

The Davis-Kahan inequality on the k-th singular vectors can be proved in exactly the same approach.

4.17 (Approximate projections) Let's add one more equivalent property to Lemma 4.1.17. Let A be an $m \times n$ matrix A with $m \geq n$, and $\varepsilon \geq 0$.

(a) Prove that A is an isometry if and only if $AA^\top$ is an orthogonal projection in $\mathbb{R}^m$ onto a subspace of dimension n.

(b) More generally, prove that that A is an ε-approximate isometry (i.e. the three equivalent properties in Lemma 4.1.17 hold) if and only if

$$\|AA^\top - P\| \leq \varepsilon$$

for some orthogonal projection P in $\mathbb{R}^m$ with $\operatorname{rank}(P) = n$.

Solution: (a) Suppose the SVD of A is $A = \sum_{i=1}^n s_i u_i v_i^\top$. A is an isometry if and only if $s_1 = \cdots = s_n = 1$. This is equivalent to $AA^\top = \sum_{i=1}^n u_i u_i^\top$, that is, $AA^\top$ is an orthogonal projection in $\mathbb{R}^m$ onto a subspace of dimension n.

(b) The spectral decomposition of A can be written as $AA^\top = \sum_{i=1}^n s_i(A)^2 u_i u_i^\top$. Let $P = \sum_{i=1}^n u_i u_i^\top$, as long as the condition $1 - \varepsilon \le s_n(A)^2 \le s_1(A)^2 \le 1 + \varepsilon$ holds, we have

$$\begin{aligned}\left\|AA^\top - P\right\| &= \left\|\sum_{i=1}^n (s_i(A)^2 - 1) u_i u_i^\top\right\| \\ &= \max_{i=1\ldots n} \left|s_i(A)^2 - 1\right| \\ &\le \varepsilon.\end{aligned}$$

On the other hand, suppose $\left\|AA^\top - P\right\| \le \varepsilon$, we have $\left|s_i(A)^2 - 1\right| \le \varepsilon$ by Weyl's inequality.

4.9 ($1 \to \infty$ and $1 \to 2$ norms) Let A be an $m \times n$ matrix. For most p and q, it is hard to express $\|A\|_{p\to q}$ in terms of the entries, but here are some exceptions.

(a) Prove that the $1 \to \infty$ norm equals the maximal absolute value of the entries:

$$\|A\|_{1\to\infty} = \max_{i,j} |A_{ij}|.$$

(b) Prove that the $1 \to 2$ norm equals the maximal Euclidean norm of the columns $A_{:j}$, and the $2 \to \infty$ norm equals the maximal Euclidean norm of the rows $A_{i:}$:

$$\|A\|_{1\to 2} = \max_{j\le n} \|A_{:j}\|_2 \quad \text{and} \quad \|A\|_{2\to\infty} = \max_{i\le m} \|A_{i:}\|_2.$$

Solution: (a) By definition of the $1 \to \infty$ norm, we have

$$\begin{aligned}\|A\|_{1\to\infty} &= \sup_{\|x\|_1\le 1} \|Ax\|_\infty \\ &= \sup_{\|x\|_1\le 1} \max_{i\le m} \left|\sum_{j=1}^n A_{ij} x_j\right| \\ &= \max_{i\le m} \sup_{\|x\|_1\le 1} \left|\sum_{j=1}^n A_{ij} x_j\right|\end{aligned}$$

$$= \max_{i \le m} \max_{j \le n} |A_{ij}|$$

$$= \max_{i \le m, j \le n} |A_{ij}| .$$

(b) Since the n-dim ℓ^1 ball B_1^n is the convex hull of $\{\pm e_1, \ldots, \pm e_n\}$, we have

$$\begin{aligned} \|A\|_{1\to 2} &= \sup_{\|x\|_1 \le 1} \|Ax\|_2 \\ &= \sup_{x \in \{\pm e_1, \ldots, \pm e_n\}} \|Ax\|_2 \\ &= \max_{j \le n} \|A_{:j}\|_2 . \end{aligned}$$

The last equation is true as $Ae_j = A_{:j}$. By duality (Exercise 4.18), we also have

$$\|A\|_{2\to\infty} = \left\|A^\top\right\|_{1\to 2} = \max_{i \le m} \left\|(A^\top)_{:i}\right\|_2 = \max_{i \le m} \|A_{i:}\|_2 .$$

4.20 ($\infty \to 1$ norm) We have seen this norm before! It was implicit in the assumption of Grothendieck inequality (Theorem 3.5.1). Let A be an $m \times n$ matrix.

(a) Prove that

$$\|A\|_{\infty\to 1} = \max_{x \in \{-1,1\}^m,\, y \in \{-1,1\}^n} |x^\top A y|. \tag{3}$$

(b) (Duality) Prove that

$$\|A\|_{\infty\to 1} = \sup_{\|Z\|_\infty \le 1, \mathrm{rank}(Z)=1} |\langle A, Z\rangle|$$

where the supremum is over all rank-one $m \times n$ matrices Z whose all entries are bounded by 1 in absolute value.

(c) Demonstrate that the identity in (4.32) does not generalize to quadratic forms: find an $n \times n$ symmetric matrix A for which

$$\max_{x \in [-1,1]^n} |x^\top A x| \neq \max_{x \in \{-1,1\}^n} |x^\top A x|.$$

Solution: (a) We have

$$\|A\|_{\infty\to 1} = \max_{\|x\|_\infty = \|y\|_\infty = 1} \left| x^\top A y \right|$$

$$= \max_{x\in\{-1,1\}^m, y\in\{-1,1\}^n} \left|x^\top A y\right|.$$

Here, the last equality holds by Exercise 3.47.

(b) Observe that

$$\begin{aligned}\|A\|_{\infty\to 1} &= \max_{\|x\|_\infty=\|y\|_\infty=1} \left|x^\top A y\right| \\ &= \max_{\|x\|_\infty=\|y\|_\infty=1} \left|\mathrm{Tr}(yx^\top A)\right| \\ &= \max_{\|x\|_\infty=\|y\|_\infty=1} \left|\langle A, xy^\top\rangle\right|.\end{aligned}$$

We claim that

$$\max_{\|x\|_\infty=\|y\|_\infty=1} \left|\langle A, xy^\top\rangle\right| = \max_{\|Z\|_\infty\le 1, \mathrm{rank}(Z)=1} \left|\langle A, xy^\top\rangle\right|$$

Indeed, for any $x\in\mathbb{R}^m$, $y\in\mathbb{R}^n$ with $\|x\|_\infty = \|y\|_\infty = 1$, $\mathrm{rank}(x^\top y) = 1$ and $\left\|x^\top y\right\| \le 1$. Thus

$$\max_{\|x\|_\infty=\|y\|_\infty=1} \left|\langle A, xy^\top\rangle\right| \le \max_{\|Z\|_\infty\le 1, \mathrm{rank}(Z)=1} \left|\langle A, xy^\top\rangle\right|$$

On the other hand, any rank-1 matrix Z can be written as $\|Z\| = \tilde{x}^\top\tilde{y}$ where $\|\tilde{x}\|_\infty = 1$. Since $\|Z\|_\infty \le 1$, we have $\|\tilde{y}\|_\infty \le 1$. Hence,

$$\max_{\|x\|_\infty=\|y\|_\infty=1} \left|\langle A, xy^\top\rangle\right| \ge \max_{\|Z\|_\infty\le 1, \mathrm{rank}(Z)=1} \left|\langle A, xy^\top\rangle\right|.$$

We have finished the proof.

(c) We can take the matrix

$$A = \begin{bmatrix} I_{n-1} & 0 \\ 0 & -1 \end{bmatrix}$$

as an example.

4.21 (Cut norm) This norm is important in theoretical computer science and graph theory. To find the cut norm of an $m\times n$ matrix A, we sum the entries of each submatrix of A, take the absolute value, and maximize over all submatrices. Formally, we define

$$\|A\|_{\mathrm{cut}} = \max_{I,J} \left|\sum_{i\in I,\, j\in J} A_{ij}\right|,$$

where the maximum is over all subsets $I \subset \{1, \dots, m\}$ and $J \subset \{1, \dots, n\}$. Prove that the cut norm is equivalent to the $\infty \to 1$ norm up to a constant factor:

$$\|A\|_{\text{cut}} \le \|A\|_{\infty \to 1} \le 4\|A\|_{\text{cut}}.$$

Solution: For any $I \subset \{1, \dots, m\}$, $J \subset \{1, \dots, n\}$,

$$\begin{aligned}
\left| \sum_{i \in I, j \in J} A_{ij} \right| &= \left| \sum_{i \le m, j \le n} A_{ij} \mathbb{1}_{\{i \in I\}} \mathbb{1}_{\{j \in J\}} \right| \\
&= \left| \sum_{i \le m, j \le n} A_{ij} \frac{(2 \cdot \mathbb{1}_{\{i \in I\}} - 1) + 1}{2} \cdot \frac{(2 \cdot \mathbb{1}_{\{j \in J\}} - 1) + 1}{2} \right| \\
&\le \max_{x \in \{-1,1\}^m, y \in \{-1,1\}^n} \left| \sum_{i \le m, j \le n} A_{ij} x_i y_j \right| \\
&\coloneqq \left| \sum_{i \le m, j \le n} A_{ij} \frac{\tilde{x}_i + 1}{2} \cdot \frac{\tilde{y}_j + 1}{2} \right|,
\end{aligned}$$

where $\tilde{x} \in \{-1, 1\}^m$ and $\tilde{y} \in \{-1, 1\}^n$. Expanding the expression gives

$$\begin{aligned}
&\left| \sum_{i \le m, j \le n} A_{ij} \frac{\tilde{x}_i + 1}{2} \cdot \frac{\tilde{y}_j + 1}{2} \right| \\
=&\frac{1}{4} \left| \sum_{i \le m, j \le n} A_{ij} \tilde{x}_i \tilde{y}_j + \sum_{i \le m, j \le n} A_{ij} + \sum_{i \le m, j \le n} A_{ij} \tilde{x}_i + \sum_{i \le m, j \le n} A_{ij} \tilde{y}_j \right| \\
\le&\frac{1}{4} \left\{ \left| \sum_{i \le m, j \le n} A_{ij} \tilde{x}_i \tilde{y}_j \right| + \left| \sum_{i \le m, j \le n} A_{ij} \right| + \left| \sum_{i \le m, j \le n} A_{ij} \tilde{x}_i \right| + \left| \sum_{i \le m, j \le n} A_{ij} \tilde{y}_j \right| \right\} \\
\le&\frac{1}{4} \cdot 4 \cdot \max_{x \in \{1,-1\}^m, y \in \{1,-1\}^n} \left| \sum_{i \le m, j \le n} A_{ij} x_i y_j \right| \\
=&\|A\|_{\infty \to 1}.
\end{aligned}$$

Therefore, $\|A\|_{\text{cut}} = \max_{I,J} \left| \sum_{i \in I, j \in J} A_{ij} \right| \le \|A\|_{\infty \to 1}$. On the other hand, for any $x \in \{-1, 1\}^m$, $y \in \{-1, 1\}^n$, let $I_0 \coloneqq \{i : x_i = 1\}$, $J_0 \coloneqq \{j : y_j = 1\}$, we can obtain

$$\left| \sum_{i \le mj \le n} A_{ij} x_i y_j \right|$$

$$
\begin{aligned}
=&\left|\sum_{i\in I_0, j\in J_0} A_{ij} - \sum_{i\in I_0, j\in J_0^c} A_{ij} - \sum_{i\in I_0^c, j\in J_0} A_{ij} + \sum_{i\in I_0^c, j\in J_0^c} A_{ij}\right| \\
\le&\left|\sum_{i\in I_0, j\in J_0} A_{ij}\right| + \left|\sum_{i\in I_0, j\in J_0^c} A_{ij}\right| + \left|\sum_{i\in I_0^c, j\in J_0} A_{ij}\right| + \left|\sum_{i\in I_0^c, j\in J_0^c} A_{ij}\right| \\
\le&4\max_{I,J}\left|\sum_{i\in I, j\in J} A_{ij}\right| \\
=&4\left\|A\right\|_{\mathrm{cut}}.
\end{aligned}
$$

We have finished the proof.

4.24 (Packing balls) In Remark 4.2.5, we noted that if $\mathcal{N}$ is an ε-separated subset of a metric space (T, d), then the closed $\varepsilon/2$-balls centered at points in $\mathcal{N}$ are disjoint.

(a) Show that the converse is false in a general metric space (T, d),

(b) but is true in any normed space $(T, \|\cdot\|)$.

Solution: (a) We consider a trivial metric space: $T = \mathbb{R}$, $d(x, y) = \mathbb{1}\{x = y\}$. Let $\mathcal{N} = \{0, 1\}$, then the 3/4-balls centered at points in $\mathcal{N}$ are disjoint, but $\mathcal{N}$ is not 3/2-separated.

(b) We proceed by contradiction. Assume the statement is false. Then there exist two points $x, y \in \mathcal{N}$ such that $B_x(\varepsilon/2) \cap B_y(\varepsilon/2) = \varnothing$ but $\|x - y\| \le \varepsilon$. Let $z = (x + y)/2$, we have $\|z - x\| = \|(y - x)/2\| = \|y - x\|/2 \le \varepsilon/2$ and $\|z - y\| = \|(x - y)/2\| = \|x - y\|/2 \le \varepsilon/2$. Therefore, $z \in B_x(\varepsilon/2) \cap B_y(\varepsilon/2)$, which leads to a contradiction.

4.25 (External covering numbers) In defining covering numbers of K, we required that the ball centers x_i lie in K. Relaxing this, the *external covering number* $\overline{\mathcal{N}}(K, d, \varepsilon)$ allows centers outside K. Prove that the usual and external covering numbers are essentially equivalent:

$$
\overline{\mathcal{N}}(K, d, \varepsilon) \le \mathcal{N}(K, d, \varepsilon) \le \overline{\mathcal{N}}(K, d, \varepsilon/2).
$$

Solution: $\overline{\mathcal{N}}(K, d, \varepsilon) \le \mathcal{N}(K, d, \varepsilon)$ holds trivially since any ε-net is also an external ε-net. Now we prove $\mathcal{N}(K, d, \varepsilon) \le \overline{\mathcal{N}}(K, d, \varepsilon/2)$. Let $\mathcal{M} = \{x_1, \ldots, x_N\}$ be an external $\varepsilon/2$-net. We assert that for any x_i, $i \in \{1, \ldots, n\}$, there exists $y_i \in K$ such that $d(x_i, y_i) \le \varepsilon/2$ (or we can remove x_i from $\mathcal{M}$). Then $\mathcal{M}' = \{y_1, \ldots y_n\}$ forms an ε-net of K. Indeed, for any $x \in K$, there exists $x_{i_0} \in \mathcal{M}$

such that $d(x_{i_0}, x) \le \varepsilon/2$. Thus

$$d(x, y_{i_0}) \le d(x, x_{i_0}) + d(x_{i_0}, y_{i_0}) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We finish the proof.

4.26 (Monotonicity of covering numbers) Although the covering numbers are monotone in ε (why?), they are not monotone in K.

(a) Show by example that, in general,

$$L \subset K \quad \not\Rightarrow \quad \mathcal{N}(L, d, \varepsilon) \le \mathcal{N}(K, d, \varepsilon).$$

(b) Prove an approximate version of monotonicity:

$$L \subset K \quad \Longrightarrow \quad \mathcal{N}(L, d, \varepsilon) \le \mathcal{N}(K, d, \varepsilon/2).$$

Solution: (a) We consider a very simple example in $\mathbb{R}$ where $d(x, y) := |x - y|$. Let $L = \{-1, 1\}$, $K = [-1, 1]$, then $\mathcal{N}(L, d, 1) = 2$, but $\mathcal{N}(L, d, 1) = 1$.

(b) It is straightforward to see that the external covering numbers $\overline{\mathcal{N}}(K, d, \varepsilon)$ are monotone in K. Therefore, by Exercise 4.25, we have for any $L \subset K$,

$$\mathcal{N}(L, d, \varepsilon) \le \overline{\mathcal{N}}(L, d, \varepsilon) \le \overline{\mathcal{N}}(K, d, \varepsilon) \le \mathcal{N}(K, d, \varepsilon/2).$$

4.27 (Volume of the Euclidean ball: a geometric argument) In this and the next two exercises, we compute the volume of the unit Euclidean ball in $\mathbb{R}^n$ in three different ways: geometric, probabilistic, and analytic. Let's start with geometric.

(a) The *canonical simplex* Δ_n consists of all points in $\mathbb{R}^n$ with nonnegative coordinates that sum to at most 1. Show that

$$\mathrm{Vol}(\Delta_n) = \frac{1}{n!}$$

(b) Recall that the unit ℓ^1 ball in $\mathbb{R}^n$ is $B_1^n := \{x \in \mathbb{R}^n : \|x\|_1 \le 1\}$. Partition it into simplices to show that

$$\mathrm{Vol}(B_1^n) = \frac{2^n}{n!} \le \left(\frac{2e}{n}\right)^n.$$

(c) Sandwich the unit Euclidean ball B_2^n between the appropriately scaled cube and the ℓ^1 ball to conclude that

$$\left(\frac{2}{\sqrt{n}}\right)^n \le \mathrm{Vol}(B_2^n) \le \left(\frac{2e}{\sqrt{n}}\right)^n.$$

So, the volume of the unit ball in high dimensions is *exponentially small*! Even though there is some slack in the bounds, they are good enough for most purposes.

Solution: (a) We proceed by induction on n. We use $\Delta_n(r)$ to denote all points in $\mathbb{R}^n$ with nonnegative coordinates that sum to at most r. Trivially, $\mathrm{Vol}(\Delta_1(r)) = r$. For $n \ge 2$, assuming $\mathrm{Vol}(\Delta_{n-1}(r)) = r^n/(n-1)!$, we have

$$\begin{aligned}
\mathrm{Vol}(\Delta_n(r)) &= \int_0^r \int_0^{1-x_1} \cdots \int_0^{1-\sum_{i=1}^{n-2} x_i} \int_0^{1-\sum_{i=1}^{n-1} x_i} 1 dx_n dx_{n-1} \ldots dx_2 dx_1 \\
&= \int_0^r \mathrm{Vol}(\Delta_{n-1}(1-x_1)) dx_1 \\
&= \int_0^r \frac{(1-x_1)^{n-1}}{(n-1)!} dx_1 \\
&= \frac{r^n}{n!}.
\end{aligned}$$

Therefore, we establish that $\mathrm{Vol}(\Delta_n(r)) = r^n/n!$.

(b) Partition $B_1^n(1)$ according to signs of n coordinates, we obtain 2^n distinct canonical simplex. Thus $\mathrm{Vol}(B_1^n(1)) = 2^n \mathrm{Vol}(\Delta_n(1)) = 2^n/n! \le (2e/n)^n$.

(c) By Cauchy's inequality, for any $x \in B_2^n(1)$,

$$\|x\|_1 = \left(\sum_{i=1}^n |x_i|\right) \le \|x\|_2 \cdot \sqrt{n} \le \sqrt{n}.$$

Therefore, $B_2^n(1) \subset B_1^n(\sqrt{n})$ and $\mathrm{Vol}(B_2^n(1)) \le B_1^n(\sqrt{n}) \le (2e/\sqrt{n})^n$. On the other hand, $B_\infty^n(1/\sqrt{n}) \subset B_2^n(1)$. Hence $\mathrm{Vol}(B_2^n(1)) \ge \mathrm{Vol}(B_\infty^n(1/\sqrt{n})) = (2/\sqrt{n})^n$.

4.28 (Volume of the Euclidean ball: a probabilistic argument) Let us tighten up the upper bound in Exercise 4.27. Deduce from the small ball probability (Exercise 3.7) that

$$\mathrm{Vol}(B_2^n) \le \left(\sqrt{\frac{2\pi e}{n}}\right)^n.$$

This bound is asymptotically sharp, as we will see from the next exercise.

Solution: Let $X \sim \mathrm{Unif}(B_\infty^n(1))$. Then the coordinates of X are independent $\mathrm{Unif}[-1,1]$

random variables. By results in Exercise 3.7, we have

$$\mathbb{P}\left(\|X\|_2 \le 1\right) \le \left(\frac{1}{2}\sqrt{\frac{2\pi e}{n}}\right)^n.$$

Since $\mathbb{P}\left(\|X\|_2 \le 1\right) = \mathrm{Vol}(B_2^n(1)) \cdot (1/2^n)$, we finish the proof.

4.29 (Volume of the Euclidean ball: an analytic argument) Finally, let us compute the volume of the Euclidean ball *exactly*.

(a) Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. Consider the unit ball of this normed space, $B := \{x \in \mathbb{R}^n : \|x\| \le 1\}$. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing, differentiable function satisfying $\lim_{x\to+\infty} f(x) = 0$. Check the following identity:

$$\int_{\mathbb{R}^n} f(\|x\|)\,dx = -\,\mathrm{Vol}(B)\int_0^\infty t^n f'(t)\,dt.$$

(b) Substitute $f(t) = e^{-t^2/2}$ and conclude that

$$\mathrm{Vol}\big(B_2^n\big) = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

(c) Denote by R_n the radius of a Euclidean ball with volume 1. Show that

$$R_n = \sqrt{\frac{n}{2\pi e}}\big(1+o(1)\big) \quad \text{as } n \to \infty.$$

The fact that R_n is large indicates that the volume of the unit ball B_2^n is small.

Solution: (a) Following similar approach as in the proof of Exercise 1.15,

$$\begin{aligned}\int_{\mathbb{R}^n} f(\|x\|)dx &= \int_{\mathbb{R}^n} -\int_{\|x\|}^\infty f'(t)dtdx \\ &= -\int_0^\infty \int_{\|x\|\le t} f'(t)dxdt \\ &= -\int_0^\infty \mathrm{Vol}(B)t^n f'(t)dt.\end{aligned}$$

We have finished the proof.

(b) Substitute $f(t) = e^{-t^2/2}$, we obtain

$$\int_{\mathbb{R}^n} e^{-\|x\|_2^2/2} dx = \mathrm{Vol}(B_2^n) \int_0^\infty t^{n+1} e^{-t^2/2} dt.$$

The conclusion follows by noting that

$$\int_{\mathbb{R}^n} e^{-\|x\|_2^2/2} dx = (2\pi)^{n/2}, \quad \int_0^\infty t^{n+1} e^{-t^2/2} dt = 2^{n/2} \Gamma\left(\frac{n}{2} + 1\right).$$

(c) Using results in part (b), we have

$$\mathrm{Vol}(B_2^n(R)) = \frac{R^n \pi^{n/2}}{\Gamma(n/2+1)}.$$

Solving the equation $\mathrm{Vol}(B_2^n(R)) = 1$ gives

$$R_n = \frac{\left(\Gamma(\frac{n}{2}+1)\right)^{\frac{1}{n}}}{\sqrt{\pi}} = \sqrt{\frac{n}{2\pi e}}(1 + o(1)).$$

The last equation holds by Stirling's approximation.

4.30 (Volume of the ℓ^p ball) Let $p \in [1, \infty]$.

(a) Repeating the geometric argument from Exercise 4.27, deduce the following bounds:

$$\left(\frac{2}{n^{1/p}}\right)^n \leq \mathrm{Vol}(B_p^n) \leq \left(\frac{2e}{n^{1/p}}\right)^n.$$

(b) Using the analytic argument from Exercise 4.29, compute the volume exactly:

$$\mathrm{Vol}(B_p^n) = \frac{(2\Gamma(1/p+1))^n}{\Gamma(n/p+1)}.$$

Solution: (a) By Holder's inequality, for any $x \in B_p^n(1)$,

$$\|x\|_1 = \left(\sum_{i=1}^n |x_i|\right) \leq \|x\|_p \cdot n^{1-1/p} \leq n^{1-1/p}.$$

Therefore, $B_p^n(1) \subset B_1^n(n^{1-1/p})$ and $\mathrm{Vol}(B_2^n(1)) \leq B_1^n(n^{1-1/p}) \leq \left(2e/n^{1/p}\right)^n$ by Exercise 4.27. On the other hand, $B_\infty^n(1/n^{1/p}) \subset B_p^n(1)$. Hence $\mathrm{Vol}(B_2^n(1)) \geq \mathrm{Vol}(B_\infty^n(1/n^{1/p})) = (2/n^{1/p})^n$.

(b) Using results in Exercise 4.29,

$$\int_{\mathbb{R}^n} e^{-\|x\|_p^p/p} dx = \mathrm{Vol}(B_p^n) \int_0^\infty t^{n+p-1} e^{-t^p/p} dt.$$

The conclusion follows by noting that

$$\int_{\mathbb{R}^n} e^{-\|x\|_p^p/p} dx = \left(2\Gamma\left(1/p+1\right) p^{1/p}\right)^n =, \quad \int_0^\infty t^{n+p-1} e^{-t^p/p} dt = p^{\frac{n}{p}} \Gamma\left(n/p+1\right).$$

4.31 (A lattice is an ε-net) The construction of an ε-net (Remark 4.2.7) is not very efficient, since nets tend to be exponentially large and hard to store. Instead, let $\mathcal{N}$ be the set of points in the scaled integer lattice $\frac{\varepsilon}{\sqrt{n}}\mathbb{Z}^n$ that lie inside the unit ball B_2^n. Check that $\mathcal{N}$ forms an ε-net of B_2^n in the Euclidean metric, and

$$|\mathcal{N}| \leq e^n \left(\frac{2}{\varepsilon} + 1\right)^n.$$

This bound nearly matches the one in Corollary 4.2.11. Also mention how to quickly approximate any given vector in the unit ball by a vector in $\mathcal{N}$.

Solution: First we define $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor)$ for $x \in \mathbb{R}^n$. For any $x \in B_2^n$, take $g(x) = \frac{\varepsilon}{\sqrt{n}} \lfloor \frac{\sqrt{n}}{\varepsilon} x \rfloor \in \mathcal{N}$. Then

$$\begin{aligned}
\|x - g(x)\|_2 &= \frac{\varepsilon}{\sqrt{n}} \left\| \frac{\sqrt{n}}{\varepsilon} x - \left\lfloor \frac{\sqrt{n}}{\varepsilon} x \right\rfloor \right\|_2 \\
&= \frac{\varepsilon}{\sqrt{n}} \sqrt{\sum_{i=1}^n \left(\left\lfloor \frac{\sqrt{n}}{\varepsilon} x_i \right\rfloor - \frac{\sqrt{n}}{\varepsilon} x_i \right)^2} \\
&\leq \frac{\varepsilon}{\sqrt{n}} \sqrt{n} \\
&\leq \varepsilon.
\end{aligned}$$

Therefore $\mathcal{N}$ is a ε-net of B_2^n. To bound the cardinality of $\mathcal{N}$, we consider the cubes $C(x) = x + \frac{\varepsilon}{2\sqrt{n}}[-1,1]^n$ for each $x \in \mathcal{N}$. These cubes are disjoint, each has volume $(\varepsilon/\sqrt{n})^n$, and they are contained in the enlargement of B_2^n, namely $B_2^n(1+\varepsilon/2)$. By a volumetric argument, we have

$$|\mathcal{N}| \leq \frac{\mathrm{vol}(B_2^n)(1+\varepsilon/2)^n}{(\varepsilon/\sqrt{n})^n}.$$

By Exercise 4.27 (c), $\mathrm{vol}(B_2^n) \leq (2e/\sqrt{n})^n$. The result follows by simplifying the fraction.

4.33 (A limitation of error correcting codes) Let us prove a converse to Theorem 4.3.5. For any error correcting code that encodes k-bit strings into n-bit strings and can correct r errors, prove that the

$$n - k \geq r \log_2 \left(\frac{n}{r} \right).$$

Solution: Let $H(x, y)$ denote the Hamming distance between two strings x and y. By definition, an error-correcting code that encodes k-bit strings into n-bit strings and corrects up to r errors induces an r-packing $\mathcal{P} \subseteq \{0,1\}^n$ with respect to the Hamming distance, with $|\mathcal{P}| \geq 2^k$. Using a volumetric argument, the sum of the volumes of these disjoint Hamming balls cannot exceed the total volume of the space:

$$\mathrm{Vol}(\{0,1\}^n) \geq |\mathcal{P}| \cdot \mathrm{Vol}(B_H^n(r)).$$

We know that $\mathrm{Vol}(\{0,1\}^n) = 2^n$, $|\mathcal{P}| \geq 2^k$, and the volume of a Hamming ball of radius r satisfies $\mathrm{Vol}(B_H^n(r)) \geq \binom{n}{r} \geq \left(\frac{n}{r}\right)^r$. Substituting these yields:

$$2^n \geq 2^k \left(\frac{n}{r} \right)^r.$$

Dividing both sides by 2^k and taking the base-2 logarithm of both sides directly yields the desired bound:

$$n - k \geq r \log_2 \left(\frac{n}{r} \right).$$

4.34 (An ε-net expansion) ε-nets help approximate vectors – here is how to use them for *exact* representation.

(a) Let $\mathcal{N}$ be an ε-net of the unit sphere S^{n-1} of $\mathbb{R}^n$ where $\varepsilon < 1$. Show that any vector $x \in S^{n-1}$ can be written as a convergent series

$$x = \sum_{k=0}^{\infty} \lambda_k x_k \quad \text{for some coefficients } 0 \leq \lambda_k \leq \varepsilon^k, \text{ and vectors } x_k \in \mathcal{N}.$$

(b) Use the ε-net expansion to give an alternative proof of Lemma 4.4.1.

Solution: **(a)** For any $x \in S^{n-1}$, there exists $x_0 \in \mathcal{N}$ such that the residual $r_1 = x - x_0$ satisfies $\|r_1\| \leq \varepsilon$. If $\|r_1\| = 0$, we have a perfect approximation and terminate the process.

Otherwise, we normalize the residual to obtain $r_1 / \|r_1\| \in S^{n-1}$. There exists $x_1 \in \mathcal{N}$ such that

the new residual satisfies:

$$\left\| \frac{r_1}{\|r_1\|} - x_1 \right\| \leq \varepsilon$$

Multiplying both sides by $\|r_1\|$ gives $\|r_1 - \|r_1\| x_1\| \leq \varepsilon \|r_1\|$. Substituting $r_1 = x - x_0$, we obtain:

$$\|x - x_0 - \|r_1\| x_1\| \leq \varepsilon \|r_1\|$$

We continue this process inductively. For any $N \in \mathbb{N}$, we construct a sequence of vectors and residuals such that:

$$\left\| x - \left(x_0 + \sum_{i=1}^{N} \prod_{j=1}^{i} \|r_j\| \, x_i \right) \right\| \leq \varepsilon \prod_{j=1}^{N} \|r_j\| \leq \varepsilon^{N+1}$$

Let $\lambda_0 = 1$ and $\lambda_k = \prod_{j=1}^{k} \|r_j\|$ for $k \geq 1$. By construction, we have $0 \leq \lambda_k \leq \varepsilon^k$. Assuming $\varepsilon \in [0, 1)$, the residual bound ε^{N+1} vanishes as $N \to \infty$. This yields the convergent series:

$$x = \sum_{k=0}^{\infty} \lambda_k x_k$$

(b) Let A be an $m \times n$ matrix, and assume $\varepsilon \in [0, 1)$. For any $x \in S^{n-1}$, we use the ε-net expansion from part (a) to write $x = \sum_{k=0}^{\infty} \lambda_k x_k$, where $x_k \in \mathcal{N}$ and $0 \leq \lambda_k \leq \varepsilon^k$. By the linearity of A and the triangle inequality, we have:

$$\|Ax\| = \left\| \sum_{k=0}^{\infty} \lambda_k A x_k \right\| \leq \sum_{k=0}^{\infty} \lambda_k \|A x_k\|$$

Since $x_k \in \mathcal{N}$ for all k, we can bound each $\|Ax_k\|$ by the supremum over the net, meaning $\|Ax_k\| \leq \sup_{y \in \mathcal{N}} \|Ay\|$. Substituting this and the upper bound for λ_k yields:

$$\|Ax\| \leq \sum_{k=0}^{\infty} \varepsilon^k \sup_{y \in \mathcal{N}} \|Ay\|$$

Because $\varepsilon < 1$, the geometric series converges to $\frac{1}{1-\varepsilon}$. Thus, we obtain:

$$\|Ax\| \leq \frac{1}{1 - \varepsilon} \sup_{y \in \mathcal{N}} \|Ay\|$$

Taking the supremum over all $x \in S^{n-1}$ on the left side completes the proof.

4.36 (Maximizing quadratic forms on an ε-net) Prove Lemma 4.4.2 by modifying the proof of Lemma 4.4.1.

Solution: The lower bound trivially holds since $\|A\| = \sup_{x\in S^{n-1}, y\in S^{m-1}} |\langle Ax, y\rangle|$. For the upper bound, let $x \in S^{n-1}, y \in S^{m-1}$ be arbitrary. There exists $x_0 \in \mathcal{N}, y \in \mathcal{M}$ such that $\|x - x_0\|_2 \leq \varepsilon, \|y - y_0\|_2 \leq \varepsilon$. Decomposing the inner product $\langle Ax, y\rangle$ and sequentially applying the triangle and Cauchy-Schwarz inequalities, we obtain:

$$\begin{aligned}
|\langle Ax, y\rangle| &= |\langle Ax_0, y_0\rangle + \langle Ax_0, y - y_0\rangle + \langle A(x - x_0), y\rangle| \\
&\leq |\langle Ax_0, y_0\rangle| + |\langle Ax_0, y - y_0\rangle| + |\langle A(x - x_0), y\rangle| \\
&\leq |\langle Ax_0, y_0\rangle| + |\langle Ax_0, y - y_0\rangle| + \left|\langle x - x_0, A^\top y\rangle\right| \\
&\leq |\langle Ax_0, y_0\rangle| + \|Ax_0\|_2 \|y - y_0\|_2 + \left\|A^\top y\right\|_2 \|x - x_0\|_2 \\
&\leq |\langle Ax_0, y_0\rangle| + 2\varepsilon \|A\|,
\end{aligned}$$

where the final inequality follows from the operator norm bounds $\|Ax\|_2 \leq \|A\|$ and $\left\|A^\top y\right\|_2 \leq \left\|A^\top\right\| = \|A\|$. Taking the supremum with respect to $x \in S^{n-1}$ and $y \in S^{m-1}$ gives

$$\|A\| = \sup_{x\in S^{n-1}, y\in S^{m-1}} |\langle Ax, y\rangle| \leq \sup_{x_0\in\mathcal{N}, y_0\in\mathcal{M}} |\langle Ax_0, y_0\rangle| + 2\varepsilon \|A\|.$$

We can obtain the final upper bound by rearranging terms.

4.37 (Deviation of the norm on an ε-net) Let A be an $m \times n$ matrix, $\mu \in \mathbb{R}$ and $\varepsilon \in [0, 1/2)$. Show that for any ε-net $\mathcal{N}$ of the sphere S^{n-1}, we have

$$\sup_{x\in S^{n-1}} \Big|\|Ax\|_2 - \mu\Big| \leq \frac{C}{1 - 2\varepsilon} \cdot \sup_{x\in\mathcal{N}} \Big|\|Ax\|_2 - \mu\Big|.$$

Solution: WLOG, we assume $\mu = 1$. Let $R = A^\top A - I_n$, then for any $x \in S^{n-1}$, $\Big|\|Ax\|_2^2 - 1\Big| = |\langle Rx, x\rangle|$. Notice that R is a symmetric $n \times n$ matrix, the operator norm of R is given by $\|R\| = \sup_{x\in S^{n-1}} |\langle Rx, x\rangle|$. For any $x \in S^{n-1}$ there exists an $x_0 \in \mathcal{N}$ such that $\|x - x_0\|_2 \leq \varepsilon$. We have:

$$|\langle Rx, x\rangle - \langle Rx_0, x_0\rangle| \leq |\langle Rx, x - x_0\rangle| + |\langle R(x - x_0), x_0\rangle| \leq 2\varepsilon\|R\|$$

Taking the supremum over all $x \in S^{n-1}$ and rearranging terms, we obtain:

$$\|R\| \leq \sup_{x_0 \in \mathcal{N}} |\langle Rx_0, x_0 \rangle| + 2\varepsilon \|R\|$$

Rearranging this expression gives $(1 - 2\varepsilon)\|R\| \leq \sup_{x_0 \in \mathcal{N}} |\langle Rx_0, x_0 \rangle|$. Translating this back into the original norm notation for the matrix A, we establish the following bound:

$$\sup_{x \in S^{n-1}} |\|Ax\|_2^2 - 1| \leq \frac{1}{1 - 2\varepsilon} \sup_{x \in \mathcal{N}} |\|Ax\|_2^2 - 1|.$$

Define $M = \sup_{x \in S^{n-1}} |\|Ax\|_2 - 1|$ and $\delta = \sup_{x \in \mathcal{N}} |\|Ax\|_2 - 1|$. Factoring the difference of squares gives $|\|Ax\|_2^2 - 1| = |\|Ax\|_2 - 1|(\|Ax\|_2 + 1)$. By the compactness of S^{n-1}, the supremum M is attained at some $x^* \in S^{n-1}$. On the ε-net, $\|Ax\|_2 \leq 1 + \delta$ for all $x \in \mathcal{N}$. Substituting these extrema into our inequality yields:

$$M(\|Ax^*\|_2 + 1) \leq \frac{1}{1 - 2\varepsilon} \delta(\delta + 2).$$

We consider two cases for $\|Ax^*\|_2$.

- Case 1: $\|Ax^*\|_2 \geq 1$. Here, $M = \|Ax^*\|_2 - 1$, so the left-hand side is $M(M + 2)$. Since the function $t \mapsto t(t + 2)$ is strictly increasing for $t \geq 0$ and $(1 - 2\varepsilon)^{-1} \geq 1$, the inequality $M(M + 2) \leq \frac{1}{1-2\varepsilon}\delta(\delta + 2)$ implies $M \leq \frac{1}{1-2\varepsilon}\delta$.
- Case 2: $\|Ax^*\|_2 < 1$. Here, $M = 1 - \|Ax^*\|_2$, yielding $0 < M \leq 1$. The left-hand side becomes $M(2 - M) \geq M$. Furthermore, since $\mathcal{N} \subset S^{n-1}$, we have $\delta \leq M \leq 1$, which bounds the right-hand side term $\delta + 2 \leq 3$. The inequality thus simplifies to $M \leq \frac{3}{1-2\varepsilon}\delta$.

Therefore, the inequality holds for all $x \in S^{n-1}$ with the absolute constant $C = 3$.

4.38 (The convex hull of an ε-net) Let's show that an ε-net of a sphere "absorbs" a slightly smaller sphere, see Figure 4.7. Let $\mathcal{N}$ be an ε-net of the unit sphere S^{n-1} for some $\varepsilon \in (0, 1)$. Prove that

$$(1 - \varepsilon)B_2^n \subset \operatorname{conv}(\mathcal{N}).$$

Here, as usual, B_2^n denotes the unit Euclidean ball in $\mathbb{R}^n$, and $\operatorname{conv}(\cdot)$ denotes the convex hull introduced in the Appetizer.

Solution: By Exercise 4.34, for any $x \in B_n^2$, we have the following expansion:

$$x = \sum_{k=0}^{\infty} \lambda_k x_k \quad \text{for some coefficients } 0 \le \lambda_k \le \varepsilon^k, \text{ and vectors } x_k \in \mathcal{N}.$$

Therefore, $(1-\varepsilon)x = \sum_{k=0}^{\infty}(1-\varepsilon)\lambda_k x_k$. Since $\mathcal{N}$ is a finite set and $\sum_{k=0}^{\infty}(1-\varepsilon)\lambda_k \le 1$, we conclude $(1-\varepsilon)x = \sum_{k=0}^{\infty}(1-\varepsilon)\lambda_k x_k \in \operatorname{conv}(\mathcal{N})$. We finish the proof.

4.39 (Random points form an ε-net) Let $g_1, \dots, g_N$ be independent $N(0, I_n)$ random vectors. For any $R, \varepsilon > 0$, show that with probability at least $1 - e^{-cn}$, the set

$$\left\{ \frac{g_1}{\sqrt{n}}, \dots, \frac{g_N}{\sqrt{n}} \right\}$$

forms an ε-net of RB_2^n (the centered ball of radius R), as long as $N \ge e^{C(R,\varepsilon)n}$. Here $C(R, \varepsilon)$ is allowed to depend only on R and ε. (Quantify this dependence!)

Solution: According to Corollary 4.2.11, $\mathcal{N}(B_2^n, \varepsilon) \le R^n(1 + \frac{2}{\varepsilon})^n$. Let $\mathcal{M}$ be an $\varepsilon/2$-net of RB_2^n, we can choose $\mathcal{M}$ such that $|\mathcal{M}| \le (1 + \frac{4R}{\varepsilon})^n$. We assume $R \ge \frac{\varepsilon}{4}$, as otherwise a single point at the origin suffices to cover the ball. By Exercise 4.23, it is sufficient to show that the random set $\{\frac{g_1}{\sqrt{n}}, \dots, \frac{g_N}{\sqrt{n}}\}$ forms an $\varepsilon/2$-net of $\mathcal{M}$.

For any fixed $x \in \mathcal{M}$, we find a ball $B(y, r)$ completely contained in $B(x, \frac{\varepsilon}{2}) \cap RB_2^n$. Let $r = \frac{\varepsilon}{4}$ and define $y = (1 - \frac{r}{R})x$. For any $z \in B(y, r)$, we have $\|z\|_2 \le \|z - y\|_2 + \|y\|_2 \le r + (1 - \frac{r}{R})\|x\|_2 \le r + (1 - \frac{r}{R})R = R$, implying $B(y, r) \subseteq RB_2^n$. Furthermore, $\|z - x\|_2 \le \|z - y\|_2 + \|y - x\|_2 \le r + \frac{r}{R}\|x\|_2 \le 2r = \frac{\varepsilon}{2}$, implying $B(y, r) \subseteq B(x, \frac{\varepsilon}{2})$.

Let p be the probability that a random vector $X = \frac{g_1}{\sqrt{n}}$ falls into $B(y, r)$. The density of the standard Gaussian vector g_1 is $(2\pi)^{-n/2}e^{-\|z\|_2^2/2}$. Since $B(y, r) \subseteq RB_2^n$, the maximum norm of any point in the scaled ball $B(\sqrt{n}y, \sqrt{n}r)$ is strictly bounded by $\sqrt{n}R$. Integrating the minimal density over the volume of the ball yields:

$$p \ge (2\pi)^{-n/2}e^{-nR^2/2}\operatorname{Vol}(B(\sqrt{n}y, \sqrt{n}r)) = (2\pi)^{-n/2}e^{-nR^2/2}(\sqrt{n}r)^n \operatorname{Vol}(B_2^n)$$

Applying the standard lower bound $\operatorname{Vol}(B_2^n) \ge \left(\frac{2}{\sqrt{n}}\right)^n$ from Exercise 4.27, we obtain:

$$p \ge (2\pi)^{-n/2}e^{-nR^2/2}(\sqrt{n}r)^n \left(\frac{2}{\sqrt{n}}\right)^n = (2\pi)^{-n/2}e^{-nR^2/2}(2r)^n$$

Substituting $r = \frac{\varepsilon}{4}$,

$$p \geq e^{-\frac{n}{2}\ln(2\pi)} e^{-nR^2/2} \left(\frac{\varepsilon}{2}\right)^n = e^{-n\beta(R,\varepsilon)},$$

where $\beta(R,\varepsilon) = \frac{R^2}{2} + \frac{\ln(2\pi)}{2} - \ln(\frac{\varepsilon}{2})$.

By the union bound, the probability that the N independent random points fail to cover $\mathcal{M}$ is at most:

$$|\mathcal{M}|(1-p)^N \leq \left(1 + \frac{4R}{\varepsilon}\right)^n e^{-pN}$$

To ensure this failure probability is bounded by e^{-cn}, it suffices to require:

$$N \geq \frac{n}{p}\left[\ln\left(1 + \frac{4R}{\varepsilon}\right) + c\right]$$

Substituting the lower bound $p \geq e^{-n\beta(R,\varepsilon)}$, this probability constraint is satisfied as long as $N \geq e^{C(R,\varepsilon)n}$, where the constant $C(R,\varepsilon)$ is any value strictly greater than $\beta(R,\varepsilon)$.

4.40 (Too many random points are not in convex positions) In Exercise 3.23, we saw that independent Gaussian random vectors $g_1, \ldots, g_N \sim N(0, I_n)$ are in convex position if $N \leq e^{cn}$. Show the converse: with probability at least $1 - e^{-cn}$, these vectors are *not* in convex position with probability if $N \geq e^{Cn}$.

Solution: By Exercise 4.39, if $N - 1 \geq e^{\beta(10,1/2)n}$, the normalized set

$$\mathcal{N} = \left\{\frac{g_1}{\sqrt{n}}, \ldots, \frac{g_{N-1}}{\sqrt{n}}\right\}$$

forms a $(1/2)$-net of $10B_2^n$ with probability at least $1 - e^{-cn}$. By Exercise 4.38, whenever $\mathcal{N}$ is a $(1/2)$-net of $10B_2^n$, we have $5B_2^n \subset \text{conv}(\mathcal{N})$.

Now consider the independent random vector $g_N \sim N(0, I_n)$. By Theorem 3.1.1, $\|g_N\|_2 \leq 5\sqrt{n}$ with probability at least $1 - e^{-c'n}$. Taking the intersection of these two independent events, which occurs with probability at least $1 - e^{-c''n}$, we find that $\frac{g_N}{\sqrt{n}} \in 5B_2^n \subset \text{conv}(\mathcal{N})$. This further yields:

$$g_N \in \text{conv}(g_1, \ldots, g_{N-1})$$

The proof is completed.

4.41 (Expected operator norm of a random matrix)

(a) Deduce from Theorem 4.4.3 that

$$\mathbb{E}\|A\| \leq CK(\sqrt{m} + \sqrt{n}).$$

(b) Deduce from Theorem 4.6.1 that

$$\mathbb{E}\left\|\frac{1}{m}A^\top A - I_n\right\| \leq CK^2\left(\sqrt{\frac{n}{m}} + \frac{n}{m}\right).$$

and

$$\sqrt{m} - CK^2\sqrt{n} \leq \mathbb{E}s_n(A) \leq \mathbb{E}s_1(A) \leq \sqrt{m} + CK^2\sqrt{n}.$$

Solution: (a) By Theorem 4.4.3, for any $t \geq 0$,

$$\mathbb{P}\left(\|A\| - CK(\sqrt{m} + \sqrt{n}) \geq t\right) \leq 2\exp(-t^2).$$

Define $X_+ = \max\left(0, \|A\| - CK(\sqrt{m} + \sqrt{n})\right)$, then for any $t \geq 0$, we also have

$$\mathbb{P}\left(X_+ \geq t\right) \leq 2\exp(-t^2).$$

Thus $\mathbb{E}X_+ = \int_0^\infty \mathbb{P}(X_+ > t)dt \leq \int_0^\infty 2e^{-t^2}dt = \sqrt{\pi}$. Therefore,

$$\mathbb{E}\left\|A\right\| \leq CK(\sqrt{m} + \sqrt{n}) + \mathbb{E}X_+ \leq CK(\sqrt{m} + \sqrt{n}) + \sqrt{\pi} \leq C'K(\sqrt{m} + \sqrt{n}).$$

(b) By Theorem 4.6.1, for any $t \geq 0$,

$$\left\|\frac{1}{m}A^\top A - I_n\right\| \leq K^2\max(\delta, \delta^2), \quad \delta = C\left(\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\right),$$

with probability at least $1 - 2\exp(-t^2)$. Since $\max(\delta, \delta^2) \leq \delta + \delta^2$, we have

$$\begin{aligned}
K^2\max(\delta, \delta^2) &\leq K^2(\delta + \delta^2) \\
&= CK^2\left(\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\right) + C^2K^2\left(\sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\right)^2 \\
&\leq CK^2\left(\sqrt{\frac{n}{m}} + \frac{n}{m}\right) + \frac{CK^2t}{\sqrt{m}} + \frac{CK^2t^2}{m},
\end{aligned}$$

where the last step uses $(a+b)^2 \leq 2a^2 + 2b^2$ and absorbs constants. Define $f(t) = CK^2(t/\sqrt{m} +$

t^2/m), so

$$\left\| \frac{1}{m} A^\top A - I_n \right\| \le CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) + f(t)$$

with probability at least $1 - 2e^{-t^2}$. Since f is increasing and maps $[0, \infty)$ onto $[0, \infty)$, the tail-sum formula with the substitution $s = f(t)$ gives

$$\begin{aligned}
\mathbb{E} \left\| \frac{1}{m} A^\top A - I_n \right\| &\le CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) + \int_0^\infty \mathbb{P} \left(\left\| \frac{1}{m} A^\top A - I_n \right\| - CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) \ge s \right) ds \\
&\le CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) + \int_0^\infty 2e^{-t^2} f'(t)\, dt \\
&= CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) + \int_0^\infty 2e^{-t^2} \cdot CK^2 \left(\frac{1}{\sqrt{m}} + \frac{2t}{m} \right) dt \\
&= CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right) + \frac{CK^2}{\sqrt{m}} \underbrace{\int_0^\infty 2e^{-t^2}\, dt}_{=\sqrt{\pi}} + \frac{CK^2}{m} \underbrace{\int_0^\infty 4te^{-t^2}\, dt}_{=2} \\
&\le C'K^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right).
\end{aligned}$$

For the singular value bounds, (4.26) gives $|s_1(A) - \sqrt{m}| \le CK^2(\sqrt{n} + t)$ with probability at least $1 - 2\exp(-t^2)$. The same argument as in part (a) yields

$$\mathbb{E} \left| s_1(A) - \sqrt{m} \right| \le CK^2\sqrt{n} + \sqrt{\pi} \le C'K^2\sqrt{n}.$$

Hence $\mathbb{E}s_1(A) \le \sqrt{m} + C'K^2\sqrt{n}$. Similarly, $\mathbb{E}\,|s_n(A) - \sqrt{m}| \le C'K^2\sqrt{n}$, which gives $\mathbb{E}s_n(A) \ge \sqrt{m} - C'K^2\sqrt{n}$. We complete the proof.

4.42 (Norm of random matrices: a lower bound) Let's prove a matching lower bound for Theorem 4.4.3. Let A be an $m \times n$ random matrix whose entries A_{ij} are independent, subgaussian random variables satisfying $\mathbb{E}A_{ij}^2 = 1$. For any $t > 0$, show that

$$\|A\| \ge \frac{1}{2}(\sqrt{m} + \sqrt{n} - t)$$

with probability at least $1 - 2\exp(-ct^2/K^4)$. Here $K = \max_{i,j} \|A_{ij}\|_{\psi_2}$.

Solution: Recall that $\|A\| \ge \|A_{1:}\|_2$ and $\|A\| \ge \|A_{:1}\|_2$, where $A_{1:}$ represents the first row of A and $A_{:1}$ represents the first column of A. We have $\|A\| \ge \frac{1}{2}(\|A_{1:}\|_2 + \|A_{:1}\|_2)$. By Theorem 3.1.1, the first row $A_{1:}$ is a random vector in $\mathbb{R}^n$ with independent sub-gaussian entries satisfying

$\mathbb{E}A_{1j}^2 = 1$, so

$$\mathbb{P}\left(\|A_{1:}\|_2 < \sqrt{n} - t\right) \le \mathbb{P}\left(\left|\|A_{1:}\|_2 - \sqrt{n}\right| \ge t\right) \le 2\exp\left(-\frac{ct^2}{K^4}\right).$$

Similarly, the first column $A_{:1}$ is a random vector in $\mathbb{R}^m$, so

$$\mathbb{P}\left(\|A_{:1}\|_2 < \sqrt{m} - t\right) \le 2\exp\left(-\frac{ct^2}{K^4}\right).$$

By a union bound,

$$\mathbb{P}\left(\|A_{1:}\|_2 \ge \sqrt{n} - t \text{ and } \|A_{:1}\|_2 \ge \sqrt{m} - t\right) \ge 1 - 2 \cdot 2\exp\left(-\frac{ct^2}{K^4}\right).$$

On this event, $\|A\| \ge \frac{1}{2}\left(\|A_{1:}\|_2 + \|A_{:1}\|_2\right) \ge \frac{1}{2}\left(\sqrt{n} - t + \sqrt{m} - t\right) = \frac{1}{2}(\sqrt{m} + \sqrt{n} - 2t)$. Replacing $2t$ by t, we obtain

$$\mathbb{P}\left(\|A\| \ge \frac{1}{2}(\sqrt{m} + \sqrt{n} - t)\right) \ge 1 - 4\exp\left(-\frac{ct^2}{4K^4}\right) \ge 1 - 2\exp\left(-\frac{c't^2}{K^4}\right).$$

This completes the proof.

4.43 (Upper bounds on subgaussian matrices: relaxing independence)

(a) Show that Theorem 4.4.3 holds without any independence assumptions if A is a subgaussian matrix (meaning that A is a subgaussian random vector in $\mathbb{R}^{m\times n}$, see Definition 3.4.1), with $K = \|A\|_{\psi_2}$.

(b) In particular, show that Theorem 4.4.3 holds if A has independent, mean zero, subgaussian rows (or columns) A_i, with $K = \max_i \|A_i\|_{\psi_2}$.

Solution: (a) **Step 1: Approximation.** Choose $\varepsilon = 1/4$. Using Corollary 4.2.11, we can find an ε-net $\mathcal{N}$ of the sphere S^{n-1} and ε-net $\mathcal{M}$ of the sphere S^{m-1} with cardinalities

$$|\mathcal{N}| \le 9^n \quad \text{and} \quad |\mathcal{M}| \le 9^m.$$

By Lemma 4.4.2, the norm of A can be bounded using these nets as follows:

$$\|A\| \le 2 \max_{x\in\mathcal{N}, y\in\mathcal{M}} |\langle Ax, y\rangle|.$$

Step 2: Concentration. Fix $x \in \mathcal{N}$ and $y \in \mathcal{M}$. Since $\mathrm{vec}(A)$ is sub-gaussian vector in $\mathbb{R}^{m\times n}$

with $\|A\|_{\psi_2} = K$, the quadratic form

$$\langle Ax, y\rangle = \langle \mathrm{vec}(A), \mathrm{vec}(xy^\top)\rangle$$

is a sub-gaussian with $\left\|\langle A, xy^\top\rangle\right\|_{\psi_2} \le K \left\|\mathrm{vec}(xy^\top)\right\|_2 = K \|x\|_2 \|y\|_2 = K$. Recalling Proposition 2.6.6(i), we can restate this as the tail bound

$$\mathbb{P}\{|\langle Ax, y\rangle| \ge u\} = \mathbb{P}\{|\langle \mathrm{vec}(A), \mathrm{vec}(xy^\top)\rangle| \ge u\} \le 2\exp(-cu^2/K^2), \quad u \ge 0.$$

Step 3: Union bound. Next, we unfix x and y using a union bound. The event $\max_{x\in\mathcal{N}, y\in\mathcal{M}} |\langle Ax, y\rangle| \ge u$ means that there exist $x \in \mathcal{N}$ and $y \in \mathcal{M}$ such that $|\langle Ax, y\rangle| \ge u$, so the union bound gives

$$\mathbb{P}\left\{\max_{x\in\mathcal{N}, y\in\mathcal{M}} |\langle Ax, y\rangle| \ge u\right\} \le \sum_{x\in\mathcal{N}, y\in\mathcal{M}} \mathbb{P}\{|\langle Ax, y\rangle| \ge u\}.$$

Using the tail bound and the estimate on the sizes of $\mathcal{N}$ and $\mathcal{M}$, we bound the probability above by

$$9^{n+m} \cdot 2\exp(-cu^2/K^2).$$

Choose

$$u = CK(\sqrt{n} + \sqrt{m} + t).$$

Then $u^2 \ge C^2K^2(n + m + t^2)$, and if the constant C is chosen sufficiently large, the exponent is large enough, say $cu^2/K^2 \ge 3(n + m) + t^2$. Thus

$$\mathbb{P}\left\{\max_{x\in\mathcal{N}, y\in\mathcal{M}} |\langle Ax, y\rangle| \ge u\right\} \le 9^{n+m} \cdot 2\exp(-3(n + m) - t^2) \le 2\exp(-t^2).$$

Finally we conclude that

$$\mathbb{P}\{\|A\| \ge 2u\} \le 2\exp(-t^2).$$

We complete the proof.

(b) Following the same argument as in part (a), we only need to show for fixed $x \in S^{n-1}$ and $y \in S^{m-1}$ that $\langle Ax, y\rangle$ is a sub-gaussian with $\left\|\langle A, xy^\top\rangle\right\|_{\psi_2} \le CK$. Since A has independent, mean zero, subgaussian rows A_i,

$$\langle Ax, y\rangle = \sum_{i=1}^{m} y_i \langle A_i, x\rangle.$$

Each term $\langle A_i, x\rangle$ is a sub-gaussian with $\|\langle A_i, x\rangle\|_{\psi_2} \le \|A_i\|_{\psi_2} \|x\|_2 \le K$. By Proposition 2.7.1,

$$\|\langle Ax, y\rangle\|_{\psi_2} = \left\|\sum_{i=1}^m y_i \langle A_i, x\rangle\right\|_{\psi_2} \le C\sqrt{\sum_{i=1}^m y_i^2 \|\langle A_i, x\rangle\|_{\psi_2}^2} \le CK.$$

We finish the proof.

4.44 (Some $p \to q$ norms of random matrices) In Theorem 4.4.3, we looked at the $2 \to 2$ operator norm of random matrices. But what about other $p \to q$ operator norms? (We defined them in Exercise 4.18 and gave some examples in Exercises 4.19, 4.20, 4.21.) Let A be an $m \times n$ matrix with independent, mean-zero, variance-one subgaussian entries A_{ij}. Denote $K = \max_{ij} \|A_{ij}\|_{\psi_2}$.

(a) Show that

$$\mathbb{E}\|A\|_{1\to\infty} \le CK\left(\sqrt{\log m} + \sqrt{\log n}\right).$$

(b) Show that

$$\mathbb{E}\|A\|_{1\to 2} \le \sqrt{m} + CK^2\sqrt{\log n}, \quad \mathbb{E}\|A\|_{2\to\infty} \le \sqrt{n} + CK^2\sqrt{\log m}.$$

Later, with more advanced tools, we will be able to handle all $p \to q$ norms (see Exercise 8.41).

(c) Let's check that the bounds in (a) and (b) are essentially optimal. If A has independent standard normal random entries, check that

$$\mathbb{E}\|A\|_{1\to\infty} \ge c\left(\sqrt{\log m} + \sqrt{\log n}\right) \quad \text{and} \quad \mathbb{E}\|A\|_{1\to 2} \ge c\left(\sqrt{m} + \sqrt{\log n}\right).$$

Solution: (a) By Exercise 4.19, $\|A\|_{1\to\infty} = \max_{i,j} |A_{ij}|$. Since A_{ij}s are independent, mean-zero, variance-one subgaussian random variables with $\|A_{ij}\|_{\psi_2} \le K$, by Proposition 2.7.6, we have

$$\mathbb{E}\|A\|_{1\to\infty} = \mathbb{E}\max_{i,j} |A_{ij}| \le CK\sqrt{\log(mn)} \le CK\left(\sqrt{\log m} + \sqrt{\log n}\right).$$

(b) By Exercise 4.19, $\|A\|_{1\to 2} = \max_{j\le n} \|A_{:j}\|_2$, where $A_{:j} \in \mathbb{R}^m$ is the j-th column of A. Using Exercise 3.13,

$$\mathbb{E}\max_{j\le n} \|A_{:j}\|_2 \le \sqrt{m} + CK^2\sqrt{\log n}.$$

Also, $\|A\|_{2\to\infty} = \max_{i\le m} \|A_{i:}\|_2$, where $A_{i:} \in \mathbb{R}^n$ is the i-th row of A. Similarly, we have

$$\mathbb{E} \max_{i\le m} \|A_{i:}\|_2 \le \sqrt{n} + CK^2\sqrt{\log m}.$$

(c) If A has independent standard normal random entries, then $\|A\|_{1\to\infty} = \max_{i,j} |A_{ij}|$ is the maximum of mn independent standard normal random variables. By Exercise 2.38, there exists an absolute constant $c > 0$ such that

$$\mathbb{E} \max_{i,j} |A_{ij}| \ge c\sqrt{\log(mn)} \ge c\left(\sqrt{\log m} + \sqrt{\log n}\right).$$

Also, $\|A\|_{1\to 2} = \max_{j\le n} \|A_{:j}\|_2$, where $A_{:j} \in \mathbb{R}^m$ is the j-th column of A. Each $\|A_{:j}\|_2$ is the norm of a standard Gaussian vector in $\mathbb{R}^m$, so by Exercise 3.13, there exists an absolute constant $c > 0$ such that

$$\mathbb{E} \max_{j\le n} \|A_{:j}\|_2 \ge \sqrt{m} + c\sqrt{\log n}.$$

This completes the proof.

4.45 (Community detection) The guarantee of the spectral clustering algorithm for community detection (Theorem 4.5.2) assumes not only that $p - q$ is not too small, but also q, the probability of across-community edges, is not too small. Let's remove the latter assumption. Design a version of the spectral clustering algorithm that results in at most $C/(p-q)^2$ misclassified vertices.

Solution: The issue in Theorem 4.5.2 is that Davis-Kahan is applied to the second eigenvalue $\lambda_2(D)$, whose spectral gap is $\delta = \min(q, \frac{p-q}{2})n$; when q is small, $\lambda_1(D) \approx \lambda_2(D)$ and δ is small. Instead of resolving individual eigenvectors, we work with the spectral projection onto the top-2 eigenspace, whose gap $\lambda_2(D) - \lambda_3(D) = \frac{p-q}{2}n$ only depends on $p - q$.

Algorithm. Compute the adjacency matrix A, then the spectral projection $P_A = \sum_{i=1}^2 v_i v_i^\top$ onto the top two eigenvectors of A, and cluster the rows of P_A into two groups.

Analysis. Recall $D = \mathbb{E}A$ has rank 2 with $\lambda_1(D) = \frac{p+q}{2}n$, $\lambda_2(D) = \frac{p-q}{2}n$, $\lambda_3(D) = \cdots = \lambda_n(D) = 0$, and unit eigenvectors $\bar{u}_1 = \frac{1}{\sqrt{n}}\mathbf{1}$, $\bar{u}_2 = \frac{1}{\sqrt{n}}(1, \ldots, 1, -1, \ldots, -1)^\top$. The spectral projection $P_D = \bar{u}_1\bar{u}_1^\top + \bar{u}_2\bar{u}_2^\top$ satisfies

$$(P_D)_{ij} = \begin{cases} 2/n & \text{if } i, j \text{ are in the same community,} \\ 0 & \text{otherwise.} \end{cases}$$

So the rows of P_D take only two values (one per community), and the squared distance between rows from different communities is $n \cdot (2/n)^2 = 4/n$.

Apply Exercise 4.13 with $k = 2$. Since $\lambda_2(D) - \lambda_3(D) = \frac{p-q}{2}n$, we get

$$\|P_A - P_D\| \leq \frac{2\,\|R\|}{\frac{p-q}{2}n}$$

where $R = A - D$. By Corollary 4.4.7, $\|R\| \leq C\sqrt{n}$ with probability at least $1 - 4e^{-n}$, so

$$\|P_A - P_D\| \leq \frac{4C}{(p-q)\sqrt{n}}.$$

Since $P_A - P_D$ has rank at most 4, by Exercise 4.4(a),

$$\|P_A - P_D\|_F^2 \leq 4\,\|P_A - P_D\|^2 \leq \frac{C'}{(p-q)^2 n}.$$

A vertex i is misclassified only if its row in P_A deviates from its row in P_D by at least half the inter-community distance, i.e., $\|(P_A)_{i,:} - (P_D)_{i,:}\|_2 \geq 1/\sqrt{n}$. Letting M be the number of misclassified vertices,

$$\frac{M}{n} \leq \sum_{i=1}^{n} \|(P_A)_{i,:} - (P_D)_{i,:}\|_2^2 = \|P_A - P_D\|_F^2 \leq \frac{C'}{(p-q)^2 n},$$

hence $M \leq C'/(p-q)^2$.

4.46 (An alternative proof of two-sided bound on random matrices) Give a simpler proof of Theorem 4.6.1, using Theorem 3.1.1 to obtain a concentration bound for $\|Ax\|_2$ and Exercise 4.37 to reduce to a union bound over a net.

Solution: Since A has independent, mean zero, subgaussian, isotropic rows A_i, for any fixed $x \in S^{n-1}$, Ax is a random vector in $\mathbb{R}^m$ with independent, mean zero, subgaussian entries satisfying $\mathbb{E}(Ax)_i^2 = 1$. By Theorem 3.1.1, there exists an absolute constant $c > 0$ such that for any $t \geq 0$,

$$\mathbb{P}\left(\left|\|Ax\|_2 - \sqrt{m}\right| \geq t\right) \leq 2\exp\left(-\frac{ct^2}{K^4}\right).$$

Here $K = \max_i \|A_i\|_{\psi_2}$. Fix $\varepsilon = 1/4$. By Corollary 4.2.11, there exists a (1/4)-net $\mathcal{N}$ of S^{n-1} with $|\mathcal{N}| \leq 9^n$. By Exercise 4.37,

$$\sup_{x \in S^{n-1}} \left|\|Ax\|_2 - \sqrt{m}\right| \leq \frac{C}{1-2\varepsilon} \max_{x \in \mathcal{N}} \left|\|Ax\|_2 - \sqrt{m}\right| = 2C \max_{x \in \mathcal{N}} \left|\|Ax\|_2 - \sqrt{m}\right|.$$

Applying a union bound over $\mathcal{N}$,

$$\mathbb{P}\left(\max_{x\in\mathcal{N}}\left|\|Ax\|_2-\sqrt{m}\right|\geq t\right)\leq 2\cdot 9^n\exp\left(-\frac{ct^2}{K^4}\right).$$

Therefore, for any $t\geq 0$,

$$\mathbb{P}\left(\sup_{x\in S^{n-1}}\left|\|Ax\|_2-\sqrt{m}\right|\geq 2Ct\right)\leq 2\exp\left(n\ln 9-\frac{ct^2}{K^4}\right).$$

Substituting $t=C_1K^2\sqrt{n}+u$ for $u\geq 0$ and choosing C_1 large enough so that $cC_1^2>\ln 9$, we obtain

$$n\ln 9-\frac{c(C_1K^2\sqrt{n}+u)^2}{K^4}\leq n\ln 9-cC_1^2n-\frac{cu^2}{K^4}\leq -\frac{cu^2}{K^4}.$$

Hence, after renaming constants,

$$\mathbb{P}\left(\sup_{x\in S^{n-1}}\left|\|Ax\|_2-\sqrt{m}\right|\geq CK^2\sqrt{n}+u\right)\leq 2\exp\left(-\frac{c'u^2}{K^4}\right),$$

which yields the two-sided singular value bound: with probability at least $1-2\exp(-c't^2/K^4)$,

$$\sqrt{m}-CK^2\sqrt{n}-t\leq s_n(A)\leq s_1(A)\leq\sqrt{m}+CK^2\sqrt{n}+t.$$

4.47 (Intermediate singular values of random matrices) Deduce from Theorem 4.6.1 the following bound on intermediate singular values. For any fixed $1\leq k\leq n$ and $t\geq 0$,

$$s_k(A)\geq\sqrt{m}-CK^2(\sqrt{k}+t)$$

with probability at least $1-2\exp(-t^2)$.

Solution: Let B denote the $m\times k$ submatrix of A formed by its first k columns. By Corollary 4.1.7,

$$s_k(A)=\max_{S:\,\dim(S)=k}\ \min_{x\in S\cap S^{n-1}}\|Ax\|_2.$$

Choosing $S_0=\operatorname{span}(e_1,\ldots,e_k)$, we obtain

$$s_k(A)\geq\min_{x\in S_0\cap S^{n-1}}\|Ax\|_2=\min_{y\in S^{k-1}}\|By\|_2=s_k(B).$$

Since the rows of A are independent, mean zero, K-subgaussian, and isotropic in $\mathbb{R}^n$, the rows of

B (their projections onto the first k coordinates) are independent, mean zero, K-subgaussian, and isotropic in $\mathbb{R}^k$. Applying Theorem 4.6.1 to the $m \times k$ matrix B (with k in place of n): for any $t \geq 0$,

$$s_k(B) \geq \sqrt{m} - CK^2(\sqrt{k} + t)$$

with probability at least $1 - 2\exp(-ct^2)$. Combining with $s_k(A) \geq s_k(B)$ and renaming constants, we conclude that for any $t \geq 0$,

$$s_k(A) \geq \sqrt{m} - CK^2(\sqrt{k} + t)$$

with probability at least $1 - 2\exp(-t^2)$, as desired.

4.48 (Covariance estimation with a relative error) Let's give a more sensitive version of covariance estimation (Theorem 4.7.1) with relative, rather than absolute, error. Show that

$$\mathbb{E} \sup_{v \in \mathbb{R}^n} \left| \frac{v^\top \Sigma_m v}{v^\top \Sigma v} - 1 \right| \leq CK^2 \left(\sqrt{\frac{n}{m}} + \frac{n}{m} \right),$$

assuming Σ is invertible. In other words, we get a uniform, relative approximation of the variance of 1D marginals of X (recall (3.6)).

Solution: Since Σ is invertible, we can write $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$, where $\Sigma^{1/2}$ is the unique positive definite square root of Σ. Let $Y = \Sigma^{-1/2}X$. Then Y is an isotropic random vector in $\mathbb{R}^n$ with $\|Y\|_{\psi_2} \leq K$. The sample covariance matrix of Y is $\Sigma'_m = \frac{1}{m}\sum_{i=1}^m Y_i Y_i^\top$, where $Y_i = \Sigma^{-1/2}X_i$ are independent copies of Y. Note that $\Sigma'_m = \Sigma^{-1/2}\Sigma_m\Sigma^{-1/2}$, so for any $v \in \mathbb{R}^n$,

$$\frac{v^\top \Sigma_m v}{v^\top \Sigma v} = \frac{v^\top \Sigma^{1/2}\Sigma'_m\Sigma^{1/2} v}{v^\top \Sigma^{1/2} I_n \Sigma^{1/2} v} = \frac{(\Sigma^{1/2}v)^\top \Sigma'_m (\Sigma^{1/2}v)}{\left\|\Sigma^{1/2}v\right\|_2^2}.$$

Thus,

$$\begin{aligned}
\sup_{v \in \mathbb{R}^n} \left| \frac{v^\top \Sigma_m v}{v^\top \Sigma v} - 1 \right| &= \sup_{v \in \mathbb{R}^n} \left| \frac{(\Sigma^{1/2}v)^\top \Sigma'_m (\Sigma^{1/2}v)}{\left\|\Sigma^{1/2}v\right\|_2^2} - 1 \right| \\
&= \sup_{u \in \mathbb{R}^n} \left| \frac{u^\top \Sigma'_m u}{\|u\|_2^2} - 1 \right| \\
&= \sup_{u \in S^{n-1}} \left| u^\top \Sigma'_m u - 1 \right| \\
&= \left\| \Sigma'_m - I_n \right\|.
\end{aligned}$$

Note that we also have

$$\|\langle Y, x\rangle\|_{\psi_2} = \left\|\langle X, \Sigma^{-1/2}x\rangle\right\|_{\psi_2} \le \left\|\langle X, \Sigma^{-1/2}x\rangle\right\|_{L_2} = \|\langle Y, x\rangle\|_{L_2}, \quad \forall x \in \mathbb{R}^n.$$

By Theorem 4.7.1, $\mathbb{E}\,\|\Sigma'_m - I_n\| \le CK^2\left(\sqrt{\frac{n}{m}} + \frac{n}{m}\right)$, which completes the proof.

4.50 (Covering numbers of low-rank matrices) An $m \times n$ matrix of rank r can be described with $(m+n+1)r$ parameters via its SVD: mr for the left singular vectors, nr for the right singular vectors, and r for the singular values. Since covering numbers are typically exponential in the dimension (see Corollary 4.2.11), we can guess that the covering numbers of the set

$$M_{m,n,r} = \{m \times n \text{ matrices } A \text{ of rank } r \text{ and } \|A\|_F \le 1\}$$

is exponential in $(m+n+1)r$. This is indeed the case!

(a) Consider the set $O_{m,r}$ consisting of $m \times r$ matrices with orthonormal columns. Prove for every $\varepsilon > 0$:

$$\mathcal{N}(O_{m,r}, \|\cdot\|_{1\to 2}, \varepsilon) \le \left(\frac{C}{\varepsilon}\right)^{mr}.$$

(b) Prove for every $\varepsilon > 0$:

$$\mathcal{N}(M_{m,n,r}, \|\cdot\|_F, \varepsilon) \le \left(\frac{C}{\varepsilon}\right)^{(m+n+1)r}.$$

(c) Conversely, show for every $\varepsilon > 0$:

$$\mathcal{N}(M_{m,n,r}, \|\cdot\|_F, \varepsilon) \ge \left(\frac{c}{\varepsilon}\right)^{\frac{1}{2}(m+n)r}.$$

Solution:

(a) Since $U \in O_{m,r}$ has orthonormal columns, each column u_j lies on S^{m-1}. The $\|\cdot\|_{1\to 2}$ norm measures the maximum column difference: for $U, \tilde{U} \in O_{m,r}$,

$$\|U - \tilde{U}\|_{1\to 2} = \max_{1\le j\le r} \|u_j - \tilde{u}_j\|_2.$$

Thus it suffices to cover each column independently on S^{m-1}. Since $O_{m,r} \subset (S^{m-1})^r$ and the $\|\cdot\|_{1\to 2}$ metric on matrices coincides with the ℓ_∞ product metric on columns, by Exercise 4.26 and

Corollary 4.2.11,

$$\mathcal{N}(O_{m,r}, \|\cdot\|_{1\to 2}, \varepsilon) \le \mathcal{N}((S^{m-1})^r, \|\cdot\|_{1\to 2}, \varepsilon/2) \le \mathcal{N}(S^{m-1}, \|\cdot\|_2, \varepsilon/2)^r \le \left(\frac{C}{\varepsilon}\right)^{mr},$$

where $C > 0$ is an absolute constant.

(b) Every $A \in M_{m,n,r}$ admits a SVD $A = U\Sigma V^T$, where $U \in O_{m,r}$, $V \in O_{n,r}$, and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_r)$ with $\sigma_i \ge 0$ and $\|\sigma\|_2 = \|A\|_F \le 1$, i.e., $\sigma = (\sigma_1, \ldots, \sigma_r) \in B_2^r \cap \mathbb{R}_+^r$. Let $\mathcal{N}_U \subset O_{m,r}$ be an $(\varepsilon/3)$-net of $O_{m,r}$ in $\|\cdot\|_{1\to 2}$, $\mathcal{N}_V \subset O_{n,r}$ an $(\varepsilon/3)$-net of $O_{n,r}$ in $\|\cdot\|_{1\to 2}$, and $\mathcal{N}_\sigma \subset B_2^r \cap \mathbb{R}_+^r$ an $(\varepsilon/3)$-net of $B_2^r \cap \mathbb{R}_+^r$ in $\|\cdot\|_2$. (All nets are chosen as subsets of their respective sets.)

For $A = U\Sigma V^T$, choose $\tilde{U} \in \mathcal{N}_U$, $\tilde{V} \in \mathcal{N}_V$, $\tilde{\sigma} \in \mathcal{N}_\sigma$ with $\|U - \tilde{U}\|_{1\to 2} \le \varepsilon/3$, $\|V - \tilde{V}\|_{1\to 2} \le \varepsilon/3$, and $\|\sigma - \tilde{\sigma}\|_2 \le \varepsilon/3$. Write $\tilde{\Sigma} = \mathrm{diag}(\tilde{\sigma})$ and $\tilde{A} = \tilde{U}\tilde{\Sigma}\tilde{V}^T$. Then

$$A - \tilde{A} = (U - \tilde{U})\Sigma V^T + \tilde{U}(\Sigma - \tilde{\Sigma})V^T + \tilde{U}\tilde{\Sigma}(V - \tilde{V})^T.$$

We bound each term in Frobenius norm. Since V has orthonormal columns, $\|(U - \tilde{U})\Sigma V^T\|_F = \|(U - \tilde{U})\Sigma\|_F$. Writing $u_i, \tilde{u}_i$ for columns of $U, \tilde{U}$:

$$\|(U - \tilde{U})\Sigma\|_F^2 = \sum_{i=1}^r \sigma_i^2 \|u_i - \tilde{u}_i\|_2^2 \le \|U - \tilde{U}\|_{1\to 2}^2 \sum_{i=1}^r \sigma_i^2 \le \left(\frac{\varepsilon}{3}\right)^2.$$

Since $\tilde{U}$ and V both have orthonormal columns, $\|\tilde{U}(\Sigma - \tilde{\Sigma})V^T\|_F = \|\Sigma - \tilde{\Sigma}\|_F = \|\sigma - \tilde{\sigma}\|_2 \le \varepsilon/3$. Since $\tilde{U} \in O_{m,r}$, we have $\|\tilde{U}\tilde{\Sigma}(V - \tilde{V})^T\|_F = \|\tilde{\Sigma}(V - \tilde{V})^T\|_F$. Writing $v_i, \tilde{v}_i$ for columns of $V, \tilde{V}$:

$$\|\tilde{\Sigma}(V - \tilde{V})^T\|_F^2 = \sum_{i=1}^r \tilde{\sigma}_i^2 \|v_i - \tilde{v}_i\|_2^2 \le \|V - \tilde{V}\|_{1\to 2}^2 \|\tilde{\sigma}\|_2^2 \le \left(\frac{\varepsilon}{3}\right)^2.$$

Here $\tilde{\sigma} \in B_2^r \cap \mathbb{R}_+^r$ implies $\|\tilde{\sigma}\|_2 \le 1$. Combining by the triangle inequality, $\|A - \tilde{A}\|_F \le \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

Thus $\{\tilde{U}\tilde{\Sigma}\tilde{V}^T : \tilde{U} \in \mathcal{N}_U, \tilde{\sigma} \in \mathcal{N}_\sigma, \tilde{V} \in \mathcal{N}_V\}$ is an ε-net for $M_{m,n,r}$. By part (a) and Corollary 4.2.11,

$$\mathcal{N}(M_{m,n,r}, \|\cdot\|_F, \varepsilon) \le |\mathcal{N}_U| \cdot |\mathcal{N}_\sigma| \cdot |\mathcal{N}_V| \le \left(\frac{C}{\varepsilon}\right)^{mr} \cdot \left(\frac{C}{\varepsilon}\right)^{r} \cdot \left(\frac{C}{\varepsilon}\right)^{nr} = \left(\frac{C}{\varepsilon}\right)^{(m+n+1)r}.$$

(c) Without loss of generality, assume $m \ge n$ (otherwise transpose). Consider the map

$\varphi\colon \mathbb{R}^{m\times r} \to \mathbb{R}^{m\times n}$ defined by $\varphi(B) = B\, I_{r,n}^T$, where $I_{r,n}$ is the $n \times r$ matrix consisting of the first r columns of I_n. For B with $\|B\|_F \le 1$, the matrix $\varphi(B)$ has rank $\le r$ and $\|\varphi(B)\|_F = \|B\|_F \le 1$, so $\varphi(B) \in M_{m,n,r}$. Moreover, $\|\varphi(B_1) - \varphi(B_2)\|_F = \|B_1 - B_2\|_F$, so φ is an isometric embedding of $B_F^{m\times r}$ into $M_{m,n,r}$.

Therefore, by Corollary 4.2.11

$$\mathcal{N}(M_{m,n,r}, \|\cdot\|_F, \varepsilon) \ge \mathcal{N}(B_F^{m\times r}, \|\cdot\|_F, \varepsilon) \ge \left(\frac{1}{2\varepsilon}\right)^{mr} \ge \left(\frac{c}{\varepsilon}\right)^{\frac{1}{2}(m+n)r},$$

where the last inequality uses $mr \ge (m+n)r/2$ (since $m \ge n$).

4.51 (Spectral clustering of the Gaussian mixture model) Prove Theorem 4.7.5, which guarantees that spectral clustering learns the Gaussian mixture model. Here is how:

(a) Compute the covariance matrix Σ of X and note that the top eigenvector u is collinear with μ.

(b) Conclude that the signs of $\langle X_i, u\rangle$ classify most points X_i correctly.

(c) Use results about covariance estimation to show that the sample covariance matrix Σ_m approximates Σ.

(d) Use the Davis-Kahan inequality (Theorem 4.1.15) to deduce that the top eigenvector $v = v_1(\Sigma_m)$ approximates u.

(e) Using (b), conclude that the signs of $\langle X_i, v\rangle$ classify most points X_i correctly.

Solution:

In the Gaussian mixture model, each data point is

$$X_i = \theta_i \mu + g_i, \quad \theta_i \overset{\text{iid}}{\sim} \text{Rademacher}, \quad g_i \overset{\text{iid}}{\sim} N(0, I_n),$$

with θ_i and g_i independent. The spectral clustering algorithm computes $\Sigma_m = \frac{1}{m}\sum_{i=1}^m X_i X_i^\top$, its top eigenvector $v = v_1(\Sigma_m)$, and classifies X_i by the sign of $\langle X_i, v\rangle$.

(a) **Population covariance.** Since θ_i and g_i are independent with $\mathbb{E}\,\theta_i = 0$ and $\mathbb{E}\, g_i = 0$:

$$\Sigma = \mathbb{E}[XX^\top] = \mu\mu^\top + I_n.$$

The rank-1 matrix $\mu\mu^\top$ has eigenvector $u := \mu/\|\mu\|_2$ with eigenvalue $\|\mu\|_2^2$, and all vectors orthogonal to μ lie in its kernel. Thus

$$\lambda_1(\Sigma) = 1 + \|\mu\|_2^2, \quad u_1(\Sigma) = \frac{\mu}{\|\mu\|_2}, \quad \lambda_2(\Sigma) = \cdots = \lambda_n(\Sigma) = 1.$$

In particular, $u = \mu/\|\mu\|_2$ is the top eigenvector of Σ, which is collinear with μ.

(b) **Classification by u.** For each data point $X_i = \theta_i \mu + g_i$:

$$\langle X_i, u\rangle = \theta_i \|\mu\|_2 + \langle g_i, u\rangle = \theta_i \|\mu\|_2 + Z_i, \quad Z_i := \langle g_i, u\rangle \sim N(0,1).$$

The point X_i is correctly classified by u if $\mathsf{sign}(\langle X_i, u\rangle) = \theta_i$. A misclassification occurs only when $|Z_i| \geq \|\mu\|_2$, which by the Gaussian tail bound has probability

$$\mathbb{P}\big(|Z_i| \geq \|\mu\|_2\big) \leq 2\, e^{-\|\mu\|_2^2/2}.$$

The expected number of misclassified points is at most $2m\, e^{-\|\mu\|_2^2/2}$.

(c) **Covariance estimation.** We verify that X satisfies condition (4.29) in Theorem 4.7.1 with an absolute constant K. For any $x \in \mathbb{R}^n$:

$$\langle X, x\rangle = \theta\langle \mu, x\rangle + \langle g, x\rangle.$$

Since $\theta\langle \mu, x\rangle$ is subgaussian with $\|\theta\langle \mu, x\rangle\|_{\psi_2} \leq C_1 |\langle \mu, x\rangle|$ and $\langle g, x\rangle \sim N(0, \|x\|_2^2)$ is subgaussian with $\|\langle g, x\rangle\|_{\psi_2} \leq C_1 \|x\|_2$, we get

$$\|\langle X, x\rangle\|_{\psi_2} \leq C_2\big(|\langle \mu, x\rangle| + \|x\|_2\big) \leq C_2\sqrt{2}\,\sqrt{\langle \mu, x\rangle^2 + \|x\|_2^2} = C_2\sqrt{2}\,\|\langle X, x\rangle\|_{L^2},$$

where the last step uses $\|\langle X, x\rangle\|_{L^2}^2 = \mathbb{E}\langle X, x\rangle^2 = x^\top \Sigma\, x = \langle \mu, x\rangle^2 + \|x\|_2^2$. So condition (4.29) holds with an absolute constant $K = C_2\sqrt{2}$.

By Theorem 4.7.1, since K is absolute, there exists an absolute constant C_3 such that

$$\mathbb{E}\,\|\Sigma_m - \Sigma\| \leq C_3 \left(\sqrt{\frac{n}{m}} + \frac{n}{m}\right) \|\Sigma\|.$$

For $m \geq Cn$ with $C \geq 1$, we have $\sqrt{n/m} + n/m \leq 2/\sqrt{C}$. By Markov's inequality, with probability

at least 0.995:

$$\|\Sigma_m - \Sigma\| \le \frac{C'}{\sqrt{C}}\,\|\Sigma\| = \frac{C'}{\sqrt{C}}\,(1 + \|\mu\|_2^2),$$

where C' is an absolute constant. We call this event $\mathcal{E}$.

(d) **Davis–Kahan perturbation.** The spectral gap of Σ around λ_1 is $\delta = \lambda_1(\Sigma) - \lambda_2(\Sigma) = \|\mu\|_2^2$. Let $v = v_1(\Sigma_m)$ be the top eigenvector of Σ_m. On the event $\mathcal{E}$, by the Davis–Kahan inequality (Theorem 4.1.15):

$$\sin\angle(u, v) \le \frac{2\,\|\Sigma_m - \Sigma\|}{\|\mu\|_2^2} \le \frac{2C'(1 + \|\mu\|_2^2)}{\sqrt{C}\,\|\mu\|_2^2} \le \frac{4C'}{\sqrt{C}},$$

where the last step uses $\|\mu\|_2 \ge C \ge 1$, so $(1+\|\mu\|_2^2)/\,\|\mu\|_2^2 \le 2$. (Note that this ratio is bounded by an absolute constant precisely because $\|\mu\|_2 \ge 1$; this is where the condition $\|\mu\|_2 \ge C$ is essential for making C independent of μ.) In particular, there exists $\theta \in \{-1, 1\}$ such that

$$\|v - \theta u\|_2^2 \le 2\sin^2\angle(u, v) \le \frac{C''}{C},$$

where $C'' = 32(C')^2$ is an absolute constant.

(e) **Classification by v.** A point X_i is misclassified by v (with the sign convention θ from part (d)) if $\mathsf{sign}(\langle X_i, v\rangle) \ne \theta\,\theta_i$. From (b), $\langle X_i, u\rangle = \theta_i\,\|\mu\|_2 + Z_i$ where $Z_i \sim N(0, 1)$. Writing $\langle X_i, v\rangle = \theta\langle X_i, u\rangle + \langle X_i, v - \theta u\rangle$, we see that misclassification requires

$$\theta\theta_i\langle X_i, v\rangle = \|\mu\|_2 + \theta_i Z_i + \theta\theta_i\langle X_i, v - \theta u\rangle \le 0,$$

which by the triangle inequality implies $|Z_i| + |\langle X_i, v - \theta u\rangle| \ge \|\mu\|_2$. Therefore, every misclassified point satisfies at least one of:

$$\text{(I) } |Z_i| \ge \|\mu\|_2/2, \qquad \text{(II) } |\langle X_i, v - \theta u\rangle| \ge \|\mu\|_2/2.$$

Condition (I). The expected number of points with $|Z_i| \ge \|\mu\|_2/2$ is at most $2m\,e^{-\|\mu\|_2^2/8}$. By Markov's inequality, the probability that more than $0.005m$ such points exist is at most $400\,e^{-\|\mu\|_2^2/8} \le 400\,e^{-C^2/8}$, which is at most 0.005 for $C \ge 10$.

Condition (II). On the event $\mathcal{E}$, by Chebyshev's inequality:

$$|\{i : |\langle X_i, v - \theta u\rangle| \ge \|\mu\|_2/2\}| \le \frac{4m}{\|\mu\|_2^2}\,(v - \theta u)^\top\Sigma_m\,(v - \theta u) \le \frac{4m\;\|\Sigma_m\|\;\|v - \theta u\|_2^2}{\|\mu\|_2^2}.$$

On the event $\mathcal{E}$, for C large enough, $\|\Sigma_m - \Sigma\| \le \|\Sigma\|$, so $\|\Sigma_m\| \le 2(1 + \|\mu\|_2^2)$. Substituting the bound on $\|v - \theta u\|_2^2$ from part (d):

$$|\{i : |\langle X_i, v - \theta u\rangle| \ge \|\mu\|_2 /2\}| \le \frac{8m(1 + \|\mu\|_2^2)}{\|\mu\|_2^2} \cdot \frac{C''}{C} \le \frac{16\, C''}{C}\, m,$$

using $(1 + \|\mu\|_2^2)/\|\mu\|_2^2 \le 2$ again. This is at most $0.005m$ when $C \ge 3200\, C''$.

Combining. We consider two events: the covariance bound $\mathcal{E}$ (failure probability ≤ 0.005), and Condition (I) (failure probability ≤ 0.005). By a union bound, with probability at least $1 - 0.005 - 0.005 = 0.99$, the total number of misclassified points is at most $0.005m + 0.005m = 0.01m$.

5 Concentration Without Independence

5.3 (Blow-up of exponentially small sets) Let A be a subset of the sphere $\sqrt{n}S^{n-1}$ such that $\sigma(A) > 2\exp(-cs^2)$.

(a) Prove that $\sigma(A_s) > 1/2$.

(b) Deduce that $\sigma(A_{2t}) \ge 1 - 2\exp(-ct^2)$ for any $t \ge s$.

Solution: Here $c > 0$ is the absolute constant from Lemma 5.1.6.

(a) Suppose $\sigma(A_s) \le 1/2$. Then $\sigma(A_s^c) \ge 1/2$, where A_s^c is the complement of A_s. By Lemma 5.1.6, we have $\sigma((A_s^c)_s) \ge 1 - 2\exp(-cs^2)$. Now we claim that $A \cap (A_s^c)_s = \emptyset$: if $x \in A$ and $x \in (A_s^c)_s$, then there exists $z \in A_s^c$ with $\|x - z\|_2 \le s$; but $x \in A$ and $\|x - z\|_2 \le s$ imply $z \in A_s$, contradicting $z \in A_s^c$. Therefore $\sigma(A) \le 1 - \sigma((A_s^c)_s) \le 2\exp(-cs^2)$, which contradicts the assumption $\sigma(A) > 2\exp(-cs^2)$.

(b) By part (a), we have $\sigma(A_t) > 1/2$ for any $t \ge s$. Then $\sigma((A_t)_t) \ge 1 - 2\exp(-ct^2)$ by Lemma 5.1.6.

5.4 (Concentration in the geodesic metric) We stated Theorem 5.1.3 for functions that are Lipschitz with respect to the Euclidean metric on the sphere. Show that it also works with the geodesic metric, which measures the shortest arc between points.

Solution: It suffices to show that the Euclidean and geodesic metrics on $\sqrt{n}S^{n-1}$ are equivalent up to a constant factor, so that the Lipschitz norms with respect to the two metrics are comparable.

For any $x, y \in \sqrt{n}S^{n-1}$, let $\theta \in [0, \pi]$ be the angle between them. Then

$$\|x - y\|_2 = 2\sqrt{n} \sin\left(\frac{\theta}{2}\right), \qquad d_g(x, y) = \sqrt{n}\,\theta.$$

Since $\frac{\theta}{2} \le \frac{\pi}{2}\sin(\frac{\theta}{2})$ for $\theta \in [0, \pi]$, we have

$$d_g(x, y) \le \frac{\pi}{2}\|x - y\|_2.$$

Now suppose f is L-Lipschitz with respect to the geodesic metric. Then for all x, y,

$$|f(x) - f(y)| \le L\, d_g(x, y) \le \frac{\pi L}{2}\|x - y\|_2,$$

so f is $\frac{\pi L}{2}$-Lipschitz with respect to the Euclidean metric. Applying Theorem 5.1.3, we get

$$\|f(X) - \mathbb{E}\, f(X)\|_{\psi_2} \le C \cdot \frac{\pi L}{2} = C'\,\|f\|_{\mathrm{Lip},\, g},$$

where $C' = C\pi/2$ is an absolute constant.

5.6 (Concentration about the mean and median are equivalent) For a random variable Z with median M, show that

$$c\|Z - \mathbb{E}Z\|_{\psi_2} \le \|Z - M\|_{\psi_2} \le C\|Z - \mathbb{E}Z\|_{\psi_2},$$

This allows us to swap the mean and median in concentration inequalities.

Solution: For the upper bound, WLOG, we can assume $\mathbb{E}Z = 0$ and $M \ge 0$ (otherwise we can consider $-Z$ instead of Z). Suppose $\|Z\|_{\psi_2} = K$, then we have

$$2 = \mathbb{E}\exp\left(\frac{Z^2}{K^2}\right) \ge \frac{1}{2}\exp\left(\frac{M^2}{K^2}\right).$$

This implies $\|M\|_{\psi_2} \le CK$ for some absolute constant C, so

$$\|Z - M\|_{\psi_2} \le \|Z\|_{\psi_2} + \|M\|_{\psi_2} \le CK.$$

For the lower bound, WLOG, we can assume $M = 0$. Suppose $\|Z\|_{\psi_2} = K$, then by centering, we

have

$$\|Z - \mathbb{E}Z\|_{\psi_2} \le CK.$$

We finish the proof.

5.7 (Concentration is equivalent to blow-up) In Section 5.1.4, we derived Lipschitz concentration on the sphere from the blow-up phenomenon (Lemma 5.1.6). Let's reverse this logic and show that Lipschitz concentration is always equivalent to blow-up. Consider a random vector X taking values in a metric space (T, d). Suppose there exists $K > 0$ such that

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \le K\|f\|_{\mathrm{Lip}}$$

for every Lipschitz function $f : T \to \mathbb{R}$. For a subset $A \subset T$, define $\sigma(A) := \mathbb{P}\{X \in A\}$, thus making σ is a probability measure on T. Show that if $\sigma(A) \ge 1/2$, then for every $t \ge 0$ we have

$$\sigma(A_t) \ge 1 - 2\exp(-ct^2/K^2).$$

Solution: By Exercise 5.6, we can replace $\mathbb{E}f(X)$ by the median M of $f(X)$ in the concentration inequality, so that

$$\|f(X) - M\|_{\psi_2} \le CK\|f\|_{\mathrm{Lip}}.$$

Now we can apply the concentration inequality to the function $d(x, A) = \inf_{y \in A} d(x, y)$. Note that here $M = 0$ since $\sigma(A) \ge 1/2$. Since $d(x, A)$ is 1-Lipschitz and $M = 0$, we have

$$\|d(X, A)\|_{\psi_2} \le CK.$$

Therefore, for any $t \ge 0$,

$$\mathbb{P}\{X \in A_t\} = \mathbb{P}\{d(X, A) \le t\} \ge 1 - 2\exp(-ct^2/K^2).$$

We finish the proof.

5.8 (Gaussian concentration) Deduce the Gaussian concentration inequality (Theorem 5.2.3) from Gaussian isoperimetric inequality (Theorem 5.2.2).

Solution: Consider a random vector $x \sim N(0, 1)$ and a 1-Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$. By

exercise 5.6, it is enough to show that

$$\|f(X) - M\|_{\psi_2} \leq C,$$

where M is defined as the median of $f(X)$. Consider the sub-level set $A = \{x \in \mathbb{R}^n : f(x) \leq M\}$. By definition of median, we have $\gamma_n(A) = \mathbb{P}\{X \in A\} \geq 1/2$. Also, define $H = \{x \in \mathbb{R}^n : x_1 \leq 0\}$. Now we can apply the Gaussian isoperimetric inequality to the set A and get

$$\gamma_n(A_t) \geq \gamma_n(H_t) = \mathbb{P}(X_1 \leq t) \geq 1 - \exp(-ct^2/2).$$

On the other hand, if $X \in A_t$, then there exists $y \in A$ with $\|X - y\|_2 \leq t$. Since $f(y) \leq M$ and f is 1-Lipschitz, we have

$$f(X) \leq f(y) + \|X - y\|_2 \leq M + t.$$

Therefore $\gamma_n(A_t) \leq \mathbb{P}\{f(X) \leq M + t\}$, and combining with the inequality above,

$$\mathbb{P}\{f(X) \leq M + t\} \geq 1 - \exp(-ct^2/2).$$

Applying the same argument to $-f$ gives

$$\mathbb{P}\{|f(X) - M| > t\} \leq 2\exp(-ct^2/2).$$

By the definition of the sub-gaussian norm, this implies $\|f(X) - M\|_{\psi_2} \leq C$. We finish the proof.

5.9 (Concentration of maximum) In Proposition 2.7.6, we estimated the expected maximum of n subgaussians. Now, let's show that the maximum concentrates.

(a) Let $X_1, \ldots, X_n$ be independent $N(0, 1)$ random variables. Prove that

$$\|\max_i X_i - \mathbb{E}\max_i X_i\|_{\psi_2} \leq C.$$

(b) More generally, let $X_1, \ldots, X_n$ be jointly normal random variables (recall Section 3.3.2), not necessarily independent. Prove that

$$\|\max_i X_i - \mathbb{E}\max_i X_i\|_{\psi_2} \leq C \max_i \sqrt{\mathrm{Var}(X_i)}.$$

Solution: (a) Consider the function $f(x) = \max_i x_i$. It is easy to verify that f is 1-Lipschitz. Then by Theorem 5.2.3, we have

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} \leq C.$$

(b) We define $X = (X_1, \ldots, X_n)$ and note that X is a jointly normal random vector. Then let $g \sim N(0, I_n)$, we can write $X = \mu + \Sigma^{1/2} g$ where $\mu = (\mathbb{E}X_1, \ldots, \mathbb{E}X_n)$ and Σ is the covariance matrix of X. Further, we use ν_i to denote the i-th row of $\Sigma^{1/2}$, so that $X_i = \mu_i + \nu_i^T g$. Consider the function $f(x) = \max_i(\mu_i + \nu_i^T x)$, which is $\max_i \|\nu_i\|_2$-Lipschitz. Then by Theorem 5.2.3, we have

$$\|\max_i X_i - \mathbb{E}\max_i X_i\|_{\psi_2} = \|f(g) - \mathbb{E}f(g)\|_{\psi_2} \leq C \max_i \|\nu_i\|_2.$$

Note that $\|\nu_i\|_2^2 = \mathrm{Var}(X_i)$, so we finish the proof.

5.10 (Concentration around the L^p norm) In Exercise 5.6 we saw how we can always swap the mean and median in concentration inequalities. Let's prove the same for the L^p norm. For a random variable Z with $\mathbb{E}Z \geq 0$ and $p \geq 1$, show that

$$\|Z - \|Z\|_{L^p}\|_{\psi_2} \leq C\sqrt{p}\|Z - \mathbb{E}Z\|_{\psi_2}.$$

Solution: Let $\mu = \mathbb{E}Z \geq 0$ and $K = \|Z - \mathbb{E}Z\|_{\psi_2}$. By the triangle inequality,

$$\|Z - \|Z\|_{L^p}\|_{\psi_2} \leq \|Z - \mu\|_{\psi_2} + |\|Z\|_{L^p} - \mu| = K + |\|Z\|_{L^p} - \mu|.$$

It remains to show $|\|Z\|_{L^p} - \mu| \leq C\sqrt{p}\,K$.

Since L^p norms are non-decreasing in p on probability spaces,

$$\|Z\|_{L^p} \geq \|Z\|_{L^1} = \mathbb{E}|Z| \geq |\mathbb{E}Z| = \mu,$$

so $\|Z\|_{L^p} - \mu \geq 0$. By Minkowski's inequality,

$$\|Z\|_{L^p} \leq \|\mu\|_{L^p} + \|Z - \mu\|_{L^p} = \mu + \|Z - \mu\|_{L^p}.$$

Since $\|Z - \mu\|_{\psi_2} = K$, the standard subgaussian moment bound gives

$$\|Z - \mu\|_{L^p} \leq CK\sqrt{p}.$$

Thus

$$\|Z\|_{L^p} - \mu \leq CK\sqrt{p}.$$

Combining the two bounds, $|\|Z\|_{L^p} - \mu| \leq CK\sqrt{p}$. Therefore,

$$\|Z - \|Z\|_{L^p}\|_{\psi_2} \leq C'(K + CK\sqrt{p}) \leq C''\sqrt{p}K,$$

where in the last step we used $1 \leq \sqrt{p}$ for $p \geq 1$. We finish the proof.

5.12 (Concentration on the continuous cube) Prove Theorem 5.2.10 for $T = [0,1]^n$ as follows:

(a) Express $X = \phi(Z)$ using Exercise 5.11. Then, use Gaussian concentration to control the deviation of $f(\phi(Z))$ in terms of $\|f \circ \phi\|_{\text{Lip}} \leq \|f\|_{\text{Lip}}\|\phi\|_{\text{Lip}}$.

(b) Show that $\|\phi\|_{\text{Lip}}$ is bounded by a constant, and then finish the proof of the theorem.

Solution:

By Exercise 5.11, if $Z \sim N(0, I_n)$ and $\phi : \mathbb{R}^n \to [0,1]^n$ is defined coordinate-wise by $\phi(z)_i = \Phi(z_i)$, where Φ is the standard normal CDF, then $X := \phi(Z)$ is uniformly distributed on $[0,1]^n$.

Now we bound $\|\phi\|_{\text{Lip}}$. For any $z, w \in \mathbb{R}^n$, by the mean value theorem applied coordinate-wise,

$$\|\phi(z) - \phi(w)\|_2^2 = \sum_{i=1}^n |\Phi(z_i) - \Phi(w_i)|^2 \leq \sum_{i=1}^n \left(\sup_{x \in \mathbb{R}} \Phi'(x)\right)^2 (z_i - w_i)^2.$$

Since $\Phi'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \leq \frac{1}{\sqrt{2\pi}}$ for all x, we obtain

$$\|\phi(z) - \phi(w)\|_2 \leq \frac{1}{\sqrt{2\pi}} \|z - w\|_2,$$

so $\|\phi\|_{\text{Lip}} \leq \frac{1}{\sqrt{2\pi}} \leq 1$.

For any 1-Lipschitz function $f : [0,1]^n \to \mathbb{R}$, the composition $f \circ \phi : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\|f \circ \phi\|_{\text{Lip}} \leq \|f\|_{\text{Lip}} \cdot \|\phi\|_{\text{Lip}} \leq 1.$$

By Theorem 5.2.3, since $Z \sim N(0, I_n)$ and $\|f \circ \phi\|_{\text{Lip}} \leq 1$,

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} = \|f(\phi(Z)) - \mathbb{E}f(\phi(Z))\|_{\psi_2} \leq C.$$

5.13 (Concentration on the continuous ball) Prove Theorem 5.2.10 for $T = \sqrt{n}B_2^n$ by a strategy similar to Exercise 5.12:

(a) Define a function $\phi : \mathbb{R}^n \to \sqrt{n}B_2^n$ that pushes forward the Gaussian measure on $\mathbb{R}^n$ into the uniform measure on $\sqrt{n}B_2^n$.

(b) Check that ϕ has bounded Lipschitz norm.

Solution:

(a) Let $Z \sim N(0, I_n)$. Write $Z = R\theta$ where $R = \|Z\|_2 \sim \chi_n$ and $\theta = Z/\|Z\|_2$ is uniform on S^{n-1}, independent of R. Let F_n denote the CDF of χ_n. Define $g(r) = \sqrt{n}\, F_n(r)^{1/n}$ and

$$\phi(z) = g(\|z\|_2) \cdot \frac{z}{\|z\|_2}.$$

Since $\phi(Z) = g(R) \cdot \theta$, the direction θ is still uniform on S^{n-1} and the radius $g(R)$ satisfies

$$\mathbb{P}(g(R) \leq r) = \mathbb{P}\Big(F_n(R) \leq \Big(\tfrac{r}{\sqrt{n}}\Big)^n\Big) = \Big(\tfrac{r}{\sqrt{n}}\Big)^n, \quad r \in [0, \sqrt{n}],$$

which is the CDF of $\|X\|_2$ for X uniformly distributed on $\sqrt{n}B_2^n$. Thus $X := \phi(Z) \sim \text{Unif}(\sqrt{n}B_2^n)$.

(b) The Jacobian of ϕ at $z \neq 0$ has eigenvalue $g'(\|z\|_2)$ in the radial direction and $g(\|z\|_2)/\|z\|_2$ in the tangential directions, so

$$\|\phi\|_{\text{Lip}} = \sup_{r>0} \max\left(g'(r),\, \frac{g(r)}{r}\right).$$

We first bound $g(r)/r$. When $r \geq \sqrt{n}$, we have $g(r)/r \leq \sqrt{n}/r \leq 1$. When $r \leq \sqrt{n}$, the Chernoff lower-tail bound gives $F_n(r) \leq (r/\sqrt{n})^n e^{(n-r^2)/2}$, so

$$\frac{g(r)}{r} \leq e^{(1-r^2/n)/2} \leq \sqrt{e}.$$

Next we bound $g'(r)$. Let f_n denote the density of χ_n, so $f_n(u) = C_n u^{n-1} e^{-u^2/2}$. For $u \leq r$, we

have $f_n(u)/f_n(r) = (u/r)^{n-1} e^{(r^2-u^2)/2} \geq (u/r)^{n-1}$, so

$$F_n(r) \geq f_n(r) \int_0^r \left(\frac{u}{r}\right)^{n-1} du = f_n(r) \cdot \frac{r}{n},$$

giving $f_n(r)/F_n(r) \leq n/r$. Therefore

$$g'(r) = \frac{g(r)}{n} \cdot \frac{f_n(r)}{F_n(r)} \leq \frac{g(r)}{r} \leq \sqrt{e}.$$

Combining the two bounds, $\|\phi\|_{\text{Lip}} \leq \sqrt{e}$.

By Theorem 5.2.3, since $Z \sim N(0, I_n)$ and $\|f \circ \phi\|_{\text{Lip}} \leq \sqrt{e}\, L$,

$$\|f(X) - \mathbb{E}f(X)\|_{\psi_2} = \|f(\phi(Z)) - \mathbb{E}f(\phi(Z))\|_{\psi_2} \leq C\sqrt{e}\, L.$$

We finish the proof.

5.14 (Johnson-Lindenstrauss with subgaussian matrices) Let A be an $m \times n$ random matrix with mean-zero, subgaussian, isotropic rows. Show that Johnson-Lindenstrauss lemma (Theorem 5.3.1) holds for $Q = (1/\sqrt{m})A$. This covers the important *binary* JL case, where A has ± 1 entries (how?)

Solution: Let the rows $A_1, \ldots, A_m$ of A be independent, mean-zero, K-subgaussian, isotropic random vectors in $\mathbb{R}^n$. Let $T = \{x_1, \ldots, x_N\} \subset \mathbb{R}^n$. We show that for $m \geq CK^4\varepsilon^{-2} \log N$, with high probability,

$$(1-\varepsilon)\|x_i - x_j\|_2 \leq \|Q(x_i - x_j)\|_2 \leq (1+\varepsilon)\|x_i - x_j\|_2$$

holds for all pairs i, j.

Step 1: Concentration for a fixed vector. Fix a unit vector $u \in S^{n-1}$. Then

$$\|Qu\|_2^2 = \frac{1}{m}\|Au\|_2^2 = \frac{1}{m}\sum_{i=1}^m \langle A_i, u\rangle^2.$$

Since each row A_i is isotropic, $\mathbb{E}\langle A_i, u\rangle^2 = u^\top \mathbb{E}[A_i^\top A_i]u = \|u\|_2^2 = 1$. Since each row is K-subgaussian, $\|\langle A_i, u\rangle\|_{\psi_2} \leq K$. Therefore $Y_i := \langle A_i, u\rangle^2 - 1$ are independent, mean-zero, subexponential random variables with $\|Y_i\|_{\psi_1} \leq C_1 K^2$.

By Bernstein's inequality (Corollary 2.8.3), for any $\varepsilon \in (0,1)$:

$$\mathbb{P}\left(\left|\|Qu\|_2^2 - 1\right| \geq \varepsilon\right) = \mathbb{P}\left(\left|\frac{1}{m}\sum_{i=1}^m Y_i\right| \geq \varepsilon\right) \leq 2\exp\left(-\frac{cm\varepsilon^2}{K^4}\right).$$

Step 2: Union bound. For each pair $i \neq j$, apply Step 1 to the unit vector $u_{ij} = (x_i - x_j)/\|x_i - x_j\|_2$. There are $\binom{N}{2} \leq N^2/2$ such pairs. By the union bound,

$$\mathbb{P}\left(\exists\, i \neq j : \left|\frac{\|Q(x_i - x_j)\|_2^2}{\|x_i - x_j\|_2^2} - 1\right| \geq \varepsilon\right) \leq N^2 \exp\left(-\frac{cm\varepsilon^2}{K^4}\right).$$

When $m \geq CK^4\varepsilon^{-2}\log N$ with C sufficiently large, we have

$$N^2 \exp\left(-\frac{cm\varepsilon^2}{K^4}\right) \leq N^2 \cdot N^{-cC} \leq N^{-(cC-2)}.$$

Choosing C so that $cC > 2$, this probability tends to zero as $N \to \infty$. Hence with high probability, for all i, j:

$$(1-\varepsilon)\|x_i - x_j\|_2^2 \leq \|Q(x_i - x_j)\|_2^2 \leq (1+\varepsilon)\|x_i - x_j\|_2^2.$$

Taking square roots, the left inequality gives $\|Q(x_i - x_j)\|_2 \geq \sqrt{1-\varepsilon}\,\|x_i - x_j\|_2 \geq (1-\varepsilon)\|x_i - x_j\|_2$, since $\sqrt{1-\varepsilon} \geq 1-\varepsilon$ for $\varepsilon \in (0,1)$. The right inequality gives $\|Q(x_i - x_j)\|_2 \leq \sqrt{1+\varepsilon}\,\|x_i - x_j\|_2 \leq (1+\varepsilon)\|x_i - x_j\|_2$, since $\sqrt{1+\varepsilon} \leq 1+\varepsilon$ for $\varepsilon \geq 0$. This yields the JL conclusion.

Binary JL: When A has i.i.d. ± 1 entries (each ± 1 with probability $1/2$), the rows are independent. Each row has mean zero. For isotropicity: $\mathbb{E}[A_i^\top A_i] = I_n$ since the entries are independent with mean 0 and variance 1. Each ± 1 entry is bounded, hence subgaussian. Therefore each row is a mean-zero, isotropic, subgaussian random vector in $\mathbb{R}^n$, and the result applies.

5.15 (Optimality of Johnson-Lindenstrauss lemma) Let's show that the dimension $n = O(\log N)$ in Johnson-Lindenstrauss lemma (Theorem 5.3.1) is optimal – even if we allow nonlinear dimension reduction maps. Here is how:

(a) Let $z_1, \ldots, z_N$ be vectors in $\mathbb{R}^n$ that satisfy $1 < \|z_i - z_j\|_2 \leq 2$ for all distinct i, j. Show that $N \leq 5^n$.

(b) Let $n < \frac{1}{2}\log N$. Find vectors $x_1, \ldots, x_N$ in $\mathbb{R}^N$ for which there does not exist any map

$T : \mathbb{R}^N \to \mathbb{R}^n$ that satisfies

$$0.99\|x_i - x_j\|_2 \le \|T(x_i) - T(x_j)\|_2 \le 1.01\|x_i - x_j\|_2 \quad \text{for all } i, j = 1, \dots, N.$$

Solution: (a) WLOG, we assume that $z_1 = 0$. Thus $\|z_i\|_2 \le 2$ for all i. Consider the balls $B(z_i, 1/2)$ of radius $1/2$ centered at z_i. Each ball is contained in $B(0, 5/2)$ since for any $z \in B(z_i, 1/2)$, we have $\|z\|_2 \le \|z - z_i\|_2 + \|z_i\|_2 \le 1/2 + 2 = 5/2$. Since $\|z_i - z_j\|_2 > 1$ for all distinct i, j, these balls are disjoint. By a standard volumetric argument, we have $N \cdot \mathrm{Vol}(B(0, 1/2)) \le \mathrm{Vol}(B(0, 5/2))$, which gives $N \le 5^n$.

(b) Consider the standard basis vectors $e_1, \dots, e_N$ in $\mathbb{R}^N$. Suppose there exists a map $T : \mathbb{R}^N \to \mathbb{R}^n$ that satisfies the distance preservation condition. Then for any distinct i, j, we have

$$0.99\sqrt{2} = 0.99\,\|e_i - e_j\| \le \|T(e_i) - T(e_j)\|_2 \le 1.01\,\|e_i - e_j\| = 1.01\sqrt{2}.$$

By the similar argument in part (a), we must have $N \le 3.05^n$. However, since $n \le \frac{1}{2}\log N$, we also have $N \ge e^{2n}$, which gives a contradiction.

5.16 (Matrix Taylor series) Let's practice with Definition 5.4.2 of matrix functions. Let X be a symmetric matrix.

(a) For any polynomial $f(x) = a_0 + a_1 x + \cdots + a_p x^p$, check that $f(X) = a_0 I_n + a_1 X + \cdots + a_p X^p$. On the right hand side, we use matrix addition and multiplication, so X^p there is interpreted as X matrix-multiplied by itself p times.

(b) More generally, if $f(x) = \sum_{k=1}^{\infty} a_k (x - x_0)^k$ is a convergent power series in a neighborhood of x_0, check that $f(X) = \sum_{k=1}^{\infty} a_k (X - x_0 I_n)^k$, where this matrix series converges in the Frobenius norm (where?), and thus also in the operator norm.

(c) Argue that

$$e^X = I_n + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots.$$

Solution: Let

$$X = \sum_{i=1}^{n} \lambda_i u_i u_i^\top = U \Lambda U^\top$$

be the spectral decomposition of X, where U is orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. By Definition

5.4.2,

$$f(X) = \sum_{i=1}^{n} f(\lambda_i) u_i u_i^\top = U f(\Lambda) U^\top.$$

(a) If $f(x) = a_0 + a_1 x + \cdots + a_p x^p$, then

$$f(\Lambda) = \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n)) = a_0 I_n + a_1 \Lambda + \cdots + a_p \Lambda^p.$$

Multiplying by U and $U^\top$, we get

$$f(X) = a_0 I_n + a_1 U \Lambda U^\top + \cdots + a_p U \Lambda^p U^\top.$$

Since $(U \Lambda U^\top)^k = U \Lambda^k U^\top$, this becomes

$$f(X) = a_0 I_n + a_1 X + \cdots + a_p X^p.$$

(b) Let R be the radius of convergence of the power series. We assume that the spectrum of X lies in the interval of convergence; in particular, it is enough to assume

$$\max_i |\lambda_i - x_0| < R.$$

For the m-th partial sum, part (a) gives

$$\sum_{k=1}^{m} a_k (X - x_0 I_n)^k = U \left(\sum_{k=1}^{m} a_k (\Lambda - x_0 I_n)^k \right) U^\top.$$

The diagonal entries of the middle matrix are

$$\sum_{k=1}^{m} a_k (\lambda_i - x_0)^k, \qquad i = 1, \ldots, n.$$

By the scalar convergence of the power series, these entries converge to $f(\lambda_i)$. Therefore the partial sums converge to

$$U \operatorname{diag}(f(\lambda_1), \ldots, f(\lambda_n)) U^\top = f(X).$$

Moreover, the convergence is in Frobenius norm, since

$$\left\| \sum_{k=1}^{m} a_k (X - x_0 I_n)^k - f(X) \right\|_F^2 = \sum_{i=1}^{n} \left| \sum_{k=1}^{m} a_k (\lambda_i - x_0)^k - f(\lambda_i) \right|^2 \to 0.$$

Since $\|A\| \leq \|A\|_F$ for every matrix A, the same convergence also holds in the operator norm.

(c) The scalar exponential function has the Taylor expansion

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

whose radius of convergence is infinity. Applying part (b), with the constant term included, gives

$$e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!} = I_n + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots .$$

5.17 (Matrix monotonicity)

(a) Check that for commuting matrices X, Y,

$$X \preceq Y \implies f(X) \preceq f(Y) \quad \text{for any weakly increasing } f : \mathbb{R} \to \mathbb{R}.$$

(b) Give an example showing that property (a) may fail for non-commuting matrices.

Solution:

(a) Since X and Y commute, they can be simultaneously diagonalized. Thus there exists an orthogonal matrix U such that

$$X = U \operatorname{diag}(x_1, \ldots, x_n) U^\top, \qquad Y = U \operatorname{diag}(y_1, \ldots, y_n) U^\top.$$

The assumption $X \preceq Y$ means $y_i - x_i \geq 0$ for all i, so $x_i \leq y_i$ for all i. Since f is weakly increasing, $f(x_i) \leq f(y_i)$ for all i. Hence

$$f(Y) - f(X) = U \operatorname{diag}(f(y_1) - f(x_1), \ldots, f(y_n) - f(x_n)) U^\top \succeq 0.$$

Thus $f(X) \preceq f(Y)$.

(b) We give a 2×2 example. Let

$$f(t) = t^3, \qquad X = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}.$$

The function $f(t) = t^3$ is increasing on $\mathbb{R}$. Also, $X \preceq Y$. However, X and Y do not commute. Now

$$X^3 = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}, \qquad Y^3 = \begin{pmatrix} -1 & -2 \\ -2 & -3 \end{pmatrix}.$$

Thus

$$f(Y) - f(X) = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}.$$

This matrix is not positive semidefinite.

5.18 (Matrix monotonicity of $1/x$ and $\ln x$) Let X and Y be symmetric $n \times n$ matrices such that X is invertible and

$$0 \preceq X \preceq Y.$$

(a) Prove that Y is also invertible, and

$$X^{-1} \succeq Y^{-1} \succeq 0.$$

(b) Check the identity

$$\ln x = \int_0^\infty \left(\frac{1}{1+t} - \frac{1}{x+t} \right) dt.$$

(c) Using the formula in (b), deduce from (a) that

$$\ln X \preceq \ln Y.$$

Solution: (a) Since X is positive semidefinite and invertible, it is positive definite. Thus $X \succeq \varepsilon I_n$ for some $\varepsilon > 0$. Since $Y \succeq X$, we also have $Y \succeq \varepsilon I_n$, so Y is also positive definite and hence invertible.

Now we consider the case $Y = I_n$. Let $X = U \operatorname{diag}(x_1, \ldots, x_n) U^\top$ be the spectral decomposition of X, where U is orthogonal and $x_i > 0$ for all i. Then $X \preceq I_n$ means $x_i \leq 1$ for all i, so $1/x_i \geq 1$

for all i. Hence

$$X^{-1} = U \operatorname{diag}(1/x_1, \ldots, 1/x_n) U^\top \succeq I_n.$$

For the general case, we first show that $X \preceq Y$ implies $Y^{-1/2} X Y^{-1/2} \preceq I_n$. Let $Y = V \operatorname{diag}(y_1, \ldots, y_n) V^\top$ be the spectral decomposition of Y, where V is orthogonal and $y_i > 0$ for all i. Then

$$Y^{-1/2} X Y^{-1/2} = VU \operatorname{diag}(x_1/y_1, \ldots, x_n/y_n) U^\top V^\top.$$

Since $X \preceq Y$ means $x_i \le y_i$ for all i, we have $x_i/y_i \le 1$ for all i, so $Y^{-1/2} X Y^{-1/2} \preceq I_n$. By the special case above, we have

$$I_n \preceq (Y^{-1/2} X Y^{-1/2})^{-1} = Y^{1/2} X^{-1} Y^{1/2}.$$

Since $Y^{1/2}$ is positive definite, we can multiply by $Y^{-1/2}$ on both sides to get $Y^{-1} \preceq X^{-1}$. (This is because if $A \succeq B$, then $CAC^\top \succeq CBC^\top$ for any matrix C.)

(b) For $x > 0$, we have

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{1+t} - \frac{1}{x+t} \right) dt &= \lim_{T \to \infty} \int_0^T \left(\frac{1}{1+t} - \frac{1}{x+t} \right) dt \\
&= \lim_{T \to \infty} \left[\ln(1+t) - \ln(x+t) \right]_0^T \\
&= \lim_{T \to \infty} \left[\ln\left(\frac{1+t}{x+t} \right) \right]_0^T \\
&= \lim_{T \to \infty} \left[\ln\left(\frac{1+T}{x+T} \right) - \ln\left(\frac{1}{x} \right) \right] \\
&= \ln(x).
\end{aligned}$$

(c) We argue that

$$\ln X = \int_0^\infty \left(\frac{1}{1+t} I_n - (X + tI_n)^{-1} \right) dt.$$

Let $X = U \operatorname{diag}(x_1, \ldots, x_n) U^\top$ be the spectral decomposition of X, where U is orthogonal and $x_i > 0$ for all i. Then

$$\begin{aligned}
\int_0^\infty \left(\frac{1}{1+t} I_n - (X + tI_n)^{-1} \right) dt &= U \operatorname{diag}\left(\int_0^\infty \left(\frac{1}{1+t} - \frac{1}{x_i + t} \right) dt \right)_{i=1}^n U^\top \\
&= U \operatorname{diag}(\ln x_1, \ldots, \ln x_n) U^\top \\
&= \ln X.
\end{aligned}$$

Now by part (a), we have $(X + tI_n)^{-1} \succeq (Y + tI_n)^{-1}$ for all $t \geq 0$. Thus

$$\frac{1}{1+t} I_n - (X + tI_n)^{-1} \preceq \frac{1}{1+t} I_n - (Y + tI_n)^{-1}$$

for all $t \geq 0$. Integrating over t gives $\ln X \preceq \ln Y$. (Note that set of positive semidefinite matrices forms a closed convex cone. Since the integral can be viewed as a limit of Riemann sums, the integral of positive semidefinite matrices is also positive semidefinite.)

5.19 (Matrix exponentiation) Let X and Y be $n \times n$ symmetric matrices.

(a) Show that if the matrices commute, i.e. $XY = YX$, then $e^{X+Y} = e^X e^Y$.

(b) Find an example of matrices X and Y such that $e^{X+Y} \neq e^X e^Y$.

Solution: (a) Since X and Y commute, they can be simultaneously diagonalized. Thus there exists an orthogonal matrix U such that

$$X = U \operatorname{diag}(x_1, \ldots, x_n) U^\top, \qquad Y = U \operatorname{diag}(y_1, \ldots, y_n) U^\top.$$

Then

$$e^{X+Y} = U \operatorname{diag}(e^{x_1+y_1}, \ldots, e^{x_n+y_n}) U^\top.$$

On the other hand,

$$e^X e^Y = U \operatorname{diag}(e^{x_1}, \ldots, e^{x_n}) U^\top U \operatorname{diag}(e^{y_1}, \ldots, e^{y_n}) U^\top = U \operatorname{diag}(e^{x_1+y_1}, \ldots, e^{x_n+y_n}) U^\top.$$

Thus $e^{X+Y} = e^X e^Y$.

(b) We give a 2×2 example. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Both matrices are symmetric. Since $X^2 = I_2$, we have

$$e^X = \cosh(1) I_2 + \sinh(1) X = \begin{pmatrix} \cosh(1) & \sinh(1) \\ \sinh(1) & \cosh(1) \end{pmatrix}.$$

Also,

$$e^Y = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}.$$

Therefore

$$e^X e^Y = \begin{pmatrix} \cosh(1) & e\sinh(1) \\ \sinh(1) & e\cosh(1) \end{pmatrix},$$

which is not symmetric. On the other hand, $X+Y$ is symmetric, so e^{X+Y} is symmetric. Hence

$$e^{X+Y} \neq e^X e^Y.$$

5.20 Deduce from matrix Bernstein inequality (Theorem 5.4.1) the following expectation bound:

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\| \lesssim \left\|\sum_{i=1}^N \mathbb{E}X_i^2\right\|^{1/2} \sqrt{1+\log n} + K(1+\log n).$$

Solution: Let

$$S = \sum_{i=1}^N X_i, \qquad \sigma^2 = \left\|\sum_{i=1}^N \mathbb{E}X_i^2\right\|.$$

By Theorem 5.4.1, for any $t \geq 0$,

$$\mathbb{P}\{\|S\| \geq t\} \leq 2n \exp\left(-\frac{t^2/2}{\sigma^2 + Kt/3}\right).$$

We first convert this into a simpler tail estimate. For $s \geq 0$, define $t_s = 2\sigma\sqrt{s} + 2Ks$, then $t_s^2 \geq 2\sigma^2 s + 2Kst_s/3$, and $\frac{t_s^2/2}{\sigma^2 + Kt_s/3} \geq s$. Therefore

$$\mathbb{P}\{\|S\| \geq t_s\} \leq 2ne^{-s}.$$

Taking $s = \log(2n) + u$, we get for every $u \geq 0$,

$$\mathbb{P}\{\|S\| \geq 2\sigma\sqrt{\log(2n)+u} + 2K(\log(2n)+u)\} \leq e^{-u}.$$

It remains to integrate this tail bound. Put

$$g(u) = 2\sigma\sqrt{\log(2n)+u} + 2K(\log(2n)+u).$$

Since g is increasing,

$$\begin{aligned}
\mathbb{E}\,\|S\| &= \int_0^\infty \mathbb{P}\{\|S\| \geq t\}\,dt \\
&\leq g(0) + \int_{g(0)}^\infty \mathbb{P}\{\|S\| \geq t\}\,dt \\
&\leq g(0) + \int_0^\infty e^{-u} g'(u)\,du \\
&\leq 2\sigma\sqrt{\log(2n)} + 2K\log(2n) + \int_0^\infty e^{-u}\left(\frac{\sigma}{\sqrt{\log(2n)+u}} + 2K\right) du \\
&\lesssim \sigma\sqrt{1+\log n} + K(1+\log n).
\end{aligned}$$

Recalling the definition of σ^2, this is exactly

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\| \lesssim \left\|\sum_{i=1}^N \mathbb{E}X_i^2\right\|^{1/2} \sqrt{1+\log n} + K(1+\log n).$$

5.21 (Matrix Hoeffding inequality) Prove matrix Hoeffding inequality (Theorem 5.4.13) following the proof of matrix Bernstein inequality (Theorem 5.4.1).

Solution: It suffices to bound the largest eigenvalue and smallest eigenvalue of $\sum_{i=1}^N \varepsilon_i A_i$. We first consider the largest eigenvalue. Fix $\lambda > 0$, by Markov's inequality,

$$\mathbb{P}\left\{\lambda_{\max}\left(\sum_{i=1}^N \varepsilon_i A_i\right) \geq t\right\} = \mathbb{P}\left\{e^{\lambda\lambda_{\max}(\sum_{i=1}^N \varepsilon_i A_i)} \geq e^{\lambda t}\right\} \leq e^{-\lambda t}\mathbb{E}\left(e^{\lambda\lambda_{\max}\left(\sum_{i=1}^N \varepsilon_i A_i\right)}\right).$$

By definition,

$$\mathbb{E}\left(e^{\lambda\lambda_{\max}\left(\sum_{i=1}^N \varepsilon_i A_i\right)}\right) = \mathbb{E}\left(\lambda_{\max}\left(e^{\lambda\sum_{i=1}^N \varepsilon_i A_i}\right)\right) \leq \mathbb{E}\left(\operatorname{tr}\left(e^{\lambda\sum_{i=1}^N \varepsilon_i A_i}\right)\right).$$

By Lieb's inequality,

$$\mathbb{E}\left(\operatorname{tr}\left(e^{\lambda\sum_{i=1}^N \varepsilon_i A_i}\right)\right) \leq \operatorname{tr}\left(\exp\left[\sum_{i=1}^N \log \mathbb{E}e^{\lambda\varepsilon_i A_i}\right]\right).$$

Now we show that $\mathbb{E}e^{\lambda\varepsilon_i A_i} \preceq e^{\lambda^2 A_i^2/2}$ for each i. Since ε_i is Rademacher, we have

$$\mathbb{E}e^{\lambda\varepsilon_i A_i} = \frac{e^{\lambda A_i} + e^{-\lambda A_i}}{2} = \sum_{k=0}^\infty \frac{\lambda^{2k} A_i^{2k}}{(2k)!} \preceq \sum_{k=0}^\infty \frac{\lambda^{2k} A_i^{2k}}{2^k k!} = e^{\lambda^2 A_i^2/2}.$$

Therefore

$$\operatorname{tr}\left(\exp\left[\sum_{i=1}^{N}\log \mathbb{E}e^{\lambda\varepsilon_i A_i}\right]\right) \le \operatorname{tr}\left(\exp\left[\sum_{i=1}^{N}\lambda^2 A_i^2/2\right]\right).$$

Since the trace of $\exp\left[\sum_{i=1}^{N}\lambda^2 A_i^2/2\right]$ is a sum of n positive eigenvalues, it is bounded by n times the maximum eigenvalue,

$$\begin{aligned}
\operatorname{tr}\left(\exp\left[\sum_{i=1}^{N}\lambda^2 A_i^2/2\right]\right) &\le n\cdot\lambda_{\max}\left(\exp\left[\sum_{i=1}^{N}\lambda^2 A_i^2/2\right]\right)\\
&= n\cdot\exp\left(\frac{\lambda^2}{2}\lambda_{\max}\left(\sum_{i=1}^{N}A_i^2\right)\right)\\
&= n\cdot\exp\left(\frac{\lambda^2}{2}\left\|\sum_{i=1}^{N}A_i^2\right\|\right)\\
&= n\cdot\exp\left(\frac{\sigma^2\lambda^2}{2}\right).
\end{aligned}$$

Here we use the fact that $\sum_{i=1}^{N}A_i^2$ is positive semidefinite. Putting everything together, we have

$$\mathbb{P}\left\{\lambda_{\max}\left(\sum_{i=1}^{N}\varepsilon_i A_i\right)\ge t\right\} \le e^{-\lambda t}\cdot n\cdot e^{\sigma^2\lambda^2/2}.$$

Optimizing over λ gives

$$\mathbb{P}\left\{\lambda_{\max}\left(\sum_{i=1}^{N}\varepsilon_i A_i\right)\ge t\right\} \le n\cdot e^{-t^2/(2\sigma^2)}.$$

Applying the same argument to $-\sum_{i=1}^{N}\varepsilon_i A_i$ yields

$$\mathbb{P}\left\{\lambda_{\min}\left(\sum_{i=1}^{N}\varepsilon_i A_i\right)\le -t\right\} \le n\cdot e^{-t^2/(2\sigma^2)}.$$

The result follows by the union bound.

5.22 (Matrix Khintchine inequality) Deduce matrix Khintchine inequality (Theorem 5.4.14) from matrix Hoeffding inequality (Theorem 5.4.13).

Solution: Combining Exercise 1.15(c) and Exercise 5.21, we have for any $p \ge 1$,

$$\mathbb{E}\left\|\sum_{i=1}^{N}\varepsilon_i A_i\right\|^p = \int_0^\infty pt^{p-1}\mathbb{P}\left\{\left\|\sum_{i=1}^{N}\varepsilon_i A_i\right\|\ge t\right\}dt$$

$$\leq \int_0^\infty pt^{p-1} \cdot 2n \cdot e^{-t^2/(2\sigma^2)} dt$$

where $\sigma^2 = \left\| \sum_{i=1}^N A_i^2 \right\|$. If $\sigma = 0$, then $\sum_i A_i^2 = 0$, and since the A_i are symmetric, this forces $A_i = 0$ for all i; the result is trivial. Thus we assume $\sigma > 0$ below. To get the right dependence on n, we use the trivial bound by 1 as well:

$$\mathbb{P}\left\{ \left\| \sum_{i=1}^N \varepsilon_i A_i \right\| \geq t \right\} \leq \min\left(1, 2ne^{-t^2/(2\sigma^2)}\right).$$

Let $L = \log(2n)$ and set

$$R = \sigma\sqrt{2(p+L)}.$$

Then

$$\begin{aligned} \mathbb{E}\left\| \sum_{i=1}^N \varepsilon_i A_i \right\|^p &\leq \int_0^R pt^{p-1}\, dt + 2n \int_R^\infty pt^{p-1} e^{-t^2/(2\sigma^2)}\, dt \\ &= R^p + 2n \cdot \frac{p}{2}(2\sigma^2)^{p/2} \int_{p+L}^\infty u^{p/2-1} e^{-u}\, du. \end{aligned}$$

We claim that, for $a \geq p$,

$$\int_a^\infty u^{p/2-1} e^{-u}\, du \leq 2a^{p/2-1} e^{-a}.$$

Indeed, if $p \geq 2$, then $a \geq p \geq 2(p/2-1)$, so for $u \geq a$, $u^{p/2-1} \leq a^{p/2-1} e^{(u-a)/2}$; if $1 \leq p < 2$, the function $u^{p/2-1}$ is decreasing and the same bound is immediate. Applying this claim with $a = p+L$, and using $2n = e^L$, we get

$$\begin{aligned} 2n \cdot \frac{p}{2}(2\sigma^2)^{p/2} \int_{p+L}^\infty u^{p/2-1} e^{-u}\, du &\leq p(2\sigma^2)^{p/2}(p+L)^{p/2-1} e^{-p} \\ &\leq (2\sigma^2(p+L))^{p/2}. \end{aligned}$$

Since $R^p = (2\sigma^2(p+L))^{p/2}$, it follows that

$$\mathbb{E}\left\| \sum_{i=1}^N \varepsilon_i A_i \right\|^p \leq 2(2\sigma^2(p+\log(2n)))^{p/2}.$$

Taking the p-th root gives

$$\left(\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i A_i\right\|^p\right)^{1/p} \le C\sqrt{p+\log n}\left\|\sum_{i=1}^N A_i^2\right\|^{1/2},$$

where $C>0$ is an absolute constant. This is the matrix Khintchine inequality.

5.23 (Matrix Bernstein inequality for rectangular matrices) Let $X_1,\dots,X_N$ be independent, mean zero, $m\times n$ random matrices, such that $\|X_i\|\le K$ almost surely for all i. Prove that for any $t\ge 0$, we have

$$\mathbb{P}\left\{\left\|\sum_{i=1}^N X_i\right\|\ge t\right\} \le 2(m+n)\exp\left(-\frac{t^2/2}{\sigma^2+Kt/3}\right),$$

where

$$\sigma^2 = \left\|\sum_{i=1}^N \mathbb{E}X_i^\top X_i\right\| + \left\|\sum_{i=1}^N \mathbb{E}X_i X_i^\top\right\|.$$

Solution: For each i, define the $(m+n)\times(m+n)$ symmetric matrix

$$Y_i = \begin{pmatrix} 0 & X_i \\ X_i^\top & 0\end{pmatrix}.$$

Then $Y_1,\dots,Y_N$ are independent, mean zero, symmetric random matrices. Moreover, $\|Y_i\| = \|X_i\|\le K$ for all i. Thus by matrix Bernstein inequality (Theorem 5.4.1), for any $t\ge 0$,

$$\mathbb{P}\left\{\left\|\sum_{i=1}^N Y_i\right\|\ge t\right\} \le 2(m+n)\exp\left(-\frac{t^2/2}{\left\|\sum_{i=1}^N \mathbb{E}Y_i^2\right\|+Kt/3}\right).$$

Since

$$\sum_{i=1}^N \mathbb{E}Y_i^2 = \begin{pmatrix} \sum_{i=1}^N \mathbb{E}X_iX_i^\top & 0 \\ 0 & \sum_{i=1}^N \mathbb{E}X_i^\top X_i\end{pmatrix},$$

we have $\left\|\sum_{i=1}^N \mathbb{E}Y_i^2\right\| = \max\left(\left\|\sum_{i=1}^N \mathbb{E}X_iX_i^\top\right\|, \left\|\sum_{i=1}^N \mathbb{E}X_i^\top X_i\right\|\right)\le\sigma^2$. Finally, note that $\left\|\sum_{i=1}^N Y_i\right\| = \left\|\sum_{i=1}^N X_i\right\|$, we finish the proof.

5.24 (Matrix Khintchine for rectangular matrices) Let's prove a version of matrix Khintchine inequality (Theorem 5.4.14) for rectangular matrices. Let $\varepsilon_1,\dots,\varepsilon_N$ be independent Rademacher

random variables and let $A_1, \dots, A_N$ be any (fixed) $m \times n$ matrices. Show that, for every $p \in [1, \infty)$ we have

$$\left(\mathbb{E} \left\| \sum_{i=1}^N \varepsilon_i A_i \right\|^p \right)^{1/p} \leq C\sigma \sqrt{p + \log(m+n)},$$

where

$$\sigma^2 = \left\| \sum_{i=1}^N A_i^\top A_i \right\| + \left\| \sum_{i=1}^N A_i A_i^\top \right\|.$$

Solution: Similar as in Exercise 5.23, we define $Y_i = \begin{pmatrix} 0 & A_i \\ A_i^\top & 0 \end{pmatrix}$ for each i. Thus by matrix Khintchine inequality (Theorem 5.4.14), for every $p \in [1, \infty)$,

$$\begin{aligned} \left(\mathbb{E} \left\| \sum_{i=1}^N \varepsilon_i A_i \right\|^p \right)^{1/p} &= \left(\mathbb{E} \left\| \sum_{i=1}^N \varepsilon_i Y_i \right\|^p \right)^{1/p} \\ &\leq C\sqrt{p + \log(m+n)} \left\| \sum_{i=1}^N Y_i^2 \right\|^{1/2}. \end{aligned}$$

Since $\left\| \sum_{i=1}^N Y_i^2 \right\| = \max\left(\left\| \sum_{i=1}^N A_i A_i^\top \right\|, \left\| \sum_{i=1}^N A_i^\top A_i \right\| \right) \leq \sigma^2$. We finish the proof.

5.25 (Stochastic block model without loops) Definition 4.5.1 of the stochastic block model $G(n, p, q)$ allows loops, meaning every vertex connects to itself with probability p. Modify this definition to disallow loops, then prove a version of Theorem 5.5.1 under this adjustment.

Solution: In the loopless stochastic block model, we can write the adjacency matrix as $A = A' - \operatorname{diag}(A')$, where A' is the adjacency matrix of the original stochastic block model with loops. Similarly, we can write the expectation of the adjacency matrix as $D = D' - pI_n$. Here $D \coloneqq \mathbb{E}A$ and $D' \coloneqq \mathbb{E}A'$. We have $\lambda_i(D) = \lambda_i(D') - p$ and $u_i(D) = u_i(D')$ for $i = 1, 2$. Therefore, we can adopt exactly the same spectral clustering algorithm as in Theorem 5.5.1, and analyze it with D in place of D'.

The rest of the proof is the same as the proof of Theorem 5.5.1. Indeed, the deterministic part is just shifted by pI_n, so the relevant eigenvectors and spectral gaps are unchanged. The random part only loses the diagonal summands; in the matrix Bernstein step this changes the variance parameter from $\max_i \sum_{j=1}^n \mathbb{E}R_{ij}^2$ to $\max_i \sum_{j \neq i} \mathbb{E}R_{ij}^2$, which is still bounded by $np = a$. Thus the same bound $\|R\| \lesssim \sqrt{a \log n}$ holds with probability at least 0.99, and the same Davis-Kahan argument gives at most $0.01n$ misclassified vertices.

5.26 (Covariance estimation: a high-probability guarantee) Prove the high-probability version of the covariance estimation result stated in Remark 5.6.5.

Solution: We adopt the same notation as in Theorem 5.6.1. Applying matrix Bernstein inequality (Theorem 5.4.1) to the sum $\sum_{i=1}^m (X_i X_i^\top - \Sigma)$, we have for any $t \geq 0$,

$$\mathbb{P}\left\{\left\|\sum_{i=1}^m (X_i X_i^\top - \Sigma)\right\| \geq t\right\} \leq 2n \exp\left(-\frac{t^2/2}{\sigma^2 + Mt/3}\right),$$

where $\sigma^2 = \left\|\sum_{i=1}^m \mathbb{E}(X_i X_i^\top - \Sigma)^2\right\|$ and M can be any upper bound on $\left\|X_i X_i^\top - \Sigma\right\|$. Using exactly the same argument as in the proof of Theorem 5.6.1, we can show that $\sigma^2 \leq K^2 m \operatorname{tr}(\Sigma) \|\Sigma\|$ and M can be chosen as $2K^2 \operatorname{tr}(\Sigma)$. If $\Sigma = 0$, the result is trivial. Otherwise let $r = \operatorname{tr}(\Sigma)/\|\Sigma\|$. Then

$$\begin{aligned}\mathbb{P}\left\{\left\|\sum_{i=1}^m (X_i X_i^\top - \Sigma)\right\| \geq t\right\} &\leq 2n \exp\left(-\frac{t^2/2}{K^2 m \operatorname{tr}(\Sigma) \|\Sigma\| + 2K^2 \operatorname{tr}(\Sigma) t/3}\right) \\ &= 2n \exp\left(-\frac{t^2/2}{K^2 m r \|\Sigma\|^2 + 2K^2 r \|\Sigma\| t/3}\right).\end{aligned}$$

Taking

$$t_0 = C\left(\sqrt{\frac{K^2 r(\log n + u)}{m}} + \frac{K^2 r(\log n + u)}{m}\right)\|\Sigma\|,$$

we have

$$\begin{aligned}\mathbb{P}\left(\|\Sigma_m - \Sigma\| \geq t_0\right) &= \mathbb{P}\left\{\left\|\sum_{i=1}^m (X_i X_i^\top - \Sigma)\right\| \geq m t_0\right\} \\ &\leq 2n \exp\left(-\frac{m^2 t_0^2/2}{K^2 m r \|\Sigma\|^2 + 2K^2 m r \|\Sigma\| t_0/3}\right).\end{aligned}$$

Since

$$m t_0 = C\left(\sqrt{K^2 m r \|\Sigma\|^2 (\log n + u)} + K^2 r \|\Sigma\| (\log n + u)\right),$$

we have, for C large enough,

$$(\log n + u)\left(K^2 m r \|\Sigma\|^2 + \frac{2K^2 r \|\Sigma\| m t_0}{3}\right) \leq \frac{m^2 t_0^2}{2}.$$

Indeed, after substituting the expression for $m t_0$, the terms on the left are dominated by the corresponding square and cross terms on the right. Therefore the exponent above is at least

$\log n + u$. Thus

$$\mathbb{P}\left(\|\Sigma_m - \Sigma\| \geq t_0\right) \leq 2ne^{-(\log n + u)} = 2e^{-u}.$$

Equivalently, with probability at least $1 - 2e^{-u}$,

$$\|\Sigma_m - \Sigma\| \leq C\left(\sqrt{\frac{K^2 r(\log n + u)}{m}} + \frac{K^2 r(\log n + u)}{m}\right)\|\Sigma\|.$$

This proves the desired high-probability guarantee.

5.27 (Covariance estimation: the boundedness assumption) Show that the boundedness assumption (5.20) cannot be dropped from Theorem 5.6.1 in general. (See Remark 5.6.6.)

Solution: We give a counterexample in the one-dimensional case, i.e. $n = 1$. Let X be a random variable such that $X = 0$ with probability $1 - 1/k^2$, and $X = \pm k$ with probability $1/2k^2$, respectively, where k is constant larger than 2. Then $\mathbb{E}X^2 = 1$, so the true covariance is $\Sigma = 1$.

Consider the event $\mathcal{E} = \{X_1 = \cdots = X_m = 0\}$. On this event, we have $\widehat{\Sigma}_m = 0$. Moreover,

$$\mathbb{P}(\mathcal{E}) = \left(1 - \frac{1}{k^2}\right)^m.$$

Choosing $k^2 \geq 2m$, we obtain

$$\mathbb{P}(\mathcal{E}) \geq 1 - \frac{m}{k^2} \geq \frac{1}{2},$$

where we used Bernoulli's inequality. Therefore, with probability at least $1/2$, $\|\widehat{\Sigma}_m - \Sigma\| = 1$. Since m is arbitrary, this shows that for every sample size m, there exists a distribution with unit covariance such that the empirical covariance has constant-order error with constant probability.

5.28 (Logarithmic factor in covariance estimation and matrix Bernstein) Let's show that a logarithmic factor is unavoidable in general covariance estimation (5.24) and matrix Bernstein inequality (5.17).

(a) Give an example of a probability distribution where the covariance estimation bound

$$\|\Sigma_m - \Sigma\| < \|\Sigma\|$$

fails with high probability unless

$$m \gtrsim n \log n.$$

(b) Conclude that the logarithmic factors in matrix Bernstein inequality (5.17) cannot be dropped.

Solution: (a) Let $X \sim \mathrm{Unif}(\sqrt{n}e_1, \dots, \sqrt{n}e_n)$. Then we have $\Sigma = I_n$. Write $X_i = \sqrt{n}e_{J_i}$, where J_i are i.i.d. uniform on $\{1, \dots, n\}$. If some coordinate j is not sampled, then $\Sigma_m e_j = 0$ and $\|\Sigma_m - \Sigma\| \geq \|(\Sigma_m - \Sigma)e_j\| = 1 = \|\Sigma\|$.

Let $I_j = \mathbf{1}\{j \text{ is not sampled}\}$ and $Z = \sum_{j=1}^n I_j$. Then Z is the number of missing coordinates, and $\mathbb{E}Z = n\left(1 - \frac{1}{n}\right)^m$. If $m \leq (1-\varepsilon)n \log n$, then

$$\mathbb{E}Z = n\left(1 - \frac{1}{n}\right)^m \geq n^{\varepsilon + o(1)} \to \infty.$$

Next,

$$\mathrm{Var}(Z) = \sum_{j=1}^n \mathrm{Var}(I_j) + 2\sum_{j<k} \mathrm{Cov}(I_j, I_k).$$

For $j \neq k$,

$$\mathbb{P}(I_j = 1, I_k = 1) = \left(1 - \frac{2}{n}\right)^m \leq \left(1 - \frac{1}{n}\right)^{2m} = \mathbb{P}(I_j = 1)\mathbb{P}(I_k = 1),$$

so $\mathrm{Cov}(I_j, I_k) \leq 0$. Therefore

$$\mathrm{Var}(Z) \leq \sum_{j=1}^n \mathrm{Var}(I_j) \leq \sum_{j=1}^n \mathbb{E}I_j = \mathbb{E}Z.$$

By Chebyshev's inequality,

$$\mathbb{P}(Z = 0) \leq \mathbb{P}\{|Z - \mathbb{E}Z| \geq \mathbb{E}Z\} \leq \frac{\mathrm{Var}(Z)}{(\mathbb{E}Z)^2} \leq \frac{1}{\mathbb{E}Z} \to 0.$$

Hence $Z \geq 1$ with high probability. Consequently, when $m \leq (1-\varepsilon)n \log n$,

$$\mathbb{P}\{\|\Sigma_m - \Sigma\| \geq \|\Sigma\|\} \to 1.$$

Thus, in general, one needs $m \gtrsim n \log n$ for $\|\Sigma_m - \Sigma\| < \|\Sigma\|$ to hold with high probability.

(b) For the same example, set $Y_i = X_i X_i^\top - I_n$. Then Y_i are independent mean-zero symmetric matrices and $\Sigma_m - I_n = m^{-1}\sum_{i=1}^m Y_i$. Also, $\|Y_i\| \leq n$, and $\left\|\sum_{i=1}^m \mathbb{E}Y_i^2\right\| \asymp mn$. Matrix Bernstein therefore gives

$$\|\Sigma_m - I_n\| \lesssim \sqrt{\frac{n \log n}{m}} + \frac{n \log n}{m}.$$

If the logarithmic factor could be removed, then one would instead get

$$\|\Sigma_m - I_n\| \lesssim \sqrt{\frac{n}{m}} + \frac{n}{m}.$$

Taking $m = Cn$ with C sufficiently large would imply $\|\Sigma_m - I_n\| < 1$ with high probability. But by part (a), for every fixed C, when $m = Cn$, some coordinate is missing with high probability, and therefore $\|\Sigma_m - I_n\| \geq 1$ with high probability. This contradiction shows that the logarithmic factor in matrix Bernstein cannot be dropped in general.

5.29 (Effective rank) For an $n \times n$ symmetric positive-semidefinite matrix Σ, we defined the *effective rank* of Σ in (5.26) as

$$r = \frac{\operatorname{tr}(\Sigma)}{\|\Sigma\|}.$$

Unlike exact rank, which counts the nonzero eigenvalues, the effective rank is a more robust sense of how many eigenvalues actually matter for the matrix's structure.

(a) Show that

$$1 \leq r(\Sigma) \leq \operatorname{rank}(\Sigma) \leq n.$$

(b) Show that this inequality is optimal: there are matrices of full rank but whose effective rank is arbitrarily close to 1.

(c) **(Stability)** Show that, unlike the algebraic rank, the effective rank is a continuous function of the matrix, e.g. with respect to the operator norm.

(d) Show that if the random vector X takes values in a k-dimensional subspace of $\mathbb{R}^n$, then $\Sigma = \mathbb{E}XX^\top$ satisfies $\operatorname{rank}(\Sigma) \leq k$, and thus $r(\Sigma) \leq k$.

Solution: (a) Trivial.

(b) We simply consider the matrix $\Sigma = \operatorname{diag}(1, \varepsilon, \varepsilon, \ldots, \varepsilon)$, where $\varepsilon > 0$ is a small constant.

(c) Let Σ and Σ' be two symmetric positive-semidefinite matrices. Then we have

$$\begin{aligned}
|r(\Sigma) - r(\Sigma')| &= \left| \frac{\operatorname{tr}(\Sigma)}{\|\Sigma\|} - \frac{\operatorname{tr}(\Sigma')}{\|\Sigma'\|} \right| \\
&= \frac{|\operatorname{tr}(\Sigma)\|\Sigma'\| - \operatorname{tr}(\Sigma')\|\Sigma\||}{\|\Sigma\|\|\Sigma'\|} \\
&\leq \frac{\operatorname{tr}(\Sigma)|\|\Sigma'\| - \|\Sigma\|| + \|\Sigma\||\operatorname{tr}(\Sigma) - \operatorname{tr}(\Sigma')|}{\|\Sigma\|\|\Sigma'\|}
\end{aligned}$$

$$\leq \frac{\operatorname{tr}(\Sigma)\|\Sigma - \Sigma'\| + \|\Sigma\|\,|\operatorname{tr}(\Sigma - \Sigma')|}{\|\Sigma\|\|\Sigma'\|}$$
$$\leq \frac{n\|\Sigma\|\|\Sigma - \Sigma'\| + \|\Sigma\| n\|\Sigma - \Sigma'\|}{\|\Sigma\|\|\Sigma'\|}$$
$$= \frac{2n\|\Sigma - \Sigma'\|}{\|\Sigma'\|}.$$

Therefore, for any fixed $\Sigma \neq 0$, suppose Σ_k is a sequence of symmetric positive-semidefinite matrices such that $\|\Sigma_k - \Sigma\| \to 0$ as $k \to \infty$. Then we have $\|\Sigma_k\| \to \|\Sigma\| \neq 0$ and thus $r(\Sigma_k) \to r(\Sigma)$ as $k \to \infty$.

(d) Since X takes values in a k-dimensional subspace, we can write $X = AY$ for some $n \times k$ matrix A and some random vector $Y \in \mathbb{R}^k$. Then $\Sigma = \mathbb{E}XX^\top = A\mathbb{E}YY^\top A^\top$. Since $\mathbb{E}YY^\top$ is a $k \times k$ positive-semidefinite matrix, it has rank at most k. Therefore, Σ has rank at most k.

5.30 (Sampling from frames) Consider an equal-norm Parseval frame $(u_1, \ldots, u_m)$ in $\mathbb{R}^n$. Show that, with high probability, a random sample of

$$m \gtrsim n \log n$$

frame elements still forms an "approximate" frame (formalize this notion).

Solution: To avoid a notational conflict, write the original frame as $(u_1, \ldots, u_N)$, and let M be the number of sampled frame elements. Since the frame is Parseval,

$$\sum_{i=1}^{N} u_i u_i^\top = I_n.$$

Equal norm and taking traces give $\|u_i\|^2 = n/N$.

Let J be uniform on $\{1, \ldots, N\}$ and set $X = \sqrt{N} u_J$. Then

$$\mathbb{E}XX^\top = I_n, \qquad \|X\|^2 = n.$$

Thus X satisfies the boundedness assumption in Theorem 5.6.1 with $\Sigma = I_n$, $K = 1$, and effective rank $r = n$. For independent samples $J_1, \ldots, J_M$, the empirical covariance is

$$\Sigma_M = \frac{1}{M} \sum_{k=1}^{M} X_k X_k^\top = \frac{N}{M} \sum_{k=1}^{M} u_{J_k} u_{J_k}^\top.$$

By the high-probability covariance estimation bound in Remark 5.6.5,

$$\|\Sigma_M - I_n\| \le C\left(\sqrt{\frac{n(\log n + u)}{M}} + \frac{n(\log n + u)}{M}\right)$$

with probability at least $1 - 2e^{-u}$. In particular, if $M \ge C\varepsilon^{-2} n \log n$, then

$$\left\| \frac{N}{M} \sum_{k=1}^{M} u_{J_k} u_{J_k}^\top - I_n \right\| \le \varepsilon$$

with high probability. This is the desired approximate frame property; equivalently, for all $x \in \mathbb{R}^n$,

$$(1-\varepsilon)\|x\|^2 \le \frac{N}{M}\sum_{k=1}^{M} \langle u_{J_k}, x\rangle^2 \le (1+\varepsilon)\|x\|^2 .$$

5.31 (Random matrices with general independent rows) Let's prove a version of Theorem 4.6.1 for random matrices with arbitrary, not necessarily subgaussian, distributions of rows. Let A be an $m \times n$ random matrix with independent, isotropic rows A_i. Assume that for some $K \ge 0$,

$$\|A_i\|_2 \le K\sqrt{n} \quad \text{almost surely for every } i.$$

Prove that, for every $t \ge 1$, one has

$$\sqrt{m} - Kt\sqrt{n \log n} \le s_n(A) \le s_1(A) \le \sqrt{m} + Kt\sqrt{n \log n}$$

with probability at least

$$1 - 2n^{-ct^2}.$$

Solution: As in the proof of Theorem 4.6.1, it suffices to control $\left\|m^{-1}A^\top A - I_n\right\|$. We write

$$\frac{1}{m}A^\top A - I_n = \frac{1}{m}\sum_{i=1}^{m}(A_i A_i^\top - I_n).$$

Since $\mathbb{E}A_i A_i^\top = I_n$ and $\|A_i\|_2 \le K\sqrt{n} = K(\mathbb{E}\|A_i\|_2^2)^{1/2}$, we can use the high-probability covariance estimation bound in Exercise 5.26 to get

$$\left\|\frac{1}{m}A^\top A - I_n\right\| \le C\left(K\sqrt{\frac{n(\log n + u)}{m}} + K^2\frac{n(\log n + u)}{m}\right)$$

with probability at least $1 - 2e^{-u}$. Let $u = t^2 \log n$. Since $t \geq 1$, we have $\log n + u \leq 2t^2 \log n$, and thus

$$\left\| \frac{1}{m} A^\top A - I_n \right\| \leq C \left(K \sqrt{\frac{t^2 n \log n}{m}} + K^2 \frac{t^2 n \log n}{m} \right)$$

with probability at least $1 - 2n^{-t^2}$. Note that the proof of Exercise 5.26 does not rely on the identical distribution of the A_i's, so the same argument applies here. The same deterministic argument as in the proof of Theorem 4.6.1 gives

$$\sqrt{m} - CKt\sqrt{n \log n} \leq s_n(A) \leq s_1(A) \leq \sqrt{m} + CKt\sqrt{n \log n}.$$

Replacing t by t/C in this bound absorbs the absolute constant into t; the probability becomes at least $1 - 2n^{-ct^2}$ for some absolute constant $c > 0$. This gives the desired bound.

5.32 (Matrix sketching) Some matrices are too big to compute eigenvalues and eigenvectors directly. A trick to handle this is subsampling–picking a random set of rows or columns to make a smaller matrix. Let's show that we can approximate the singular values of a tall $N \times n$ matrix A using a smaller matrix B, made by randomly selecting

$$m = O(n \log n)$$

rows (with replacement, uniform probability). Assuming all rows of A have the same norm, prove that if $m \geq Cn \log n$, then the singular values satisfy

$$\max_{i=1,\dots,n} \left| s_i(A)^2 - \frac{N}{m} s_i(B)^2 \right| \leq 0.1\, s_1(A)^2$$

with probability at least 0.9.

Solution: Let A_i be the i-th row of A. We can write $B = (A_{J_1}^\top, \dots, A_{J_m}^\top)^\top$ where $J_1, \dots, J_m$ are independent and uniformly distributed on $\{1, \dots, N\}$. Then we have

$$B^\top B = \sum_{k=1}^m A_{J_k} A_{J_k}^\top, \quad \mathbb{E} B^\top B = \frac{m}{N} \sum_{i=1}^N A_i A_i^\top = \frac{m}{N} A^\top A.$$

Since all rows of A have the same norm, $\|A_{J_i}\|_2 \leq (\mathbb{E}\, \|A_{J_i}\|_2^2)^{1/2} = \|A_1\|_2$ for each i. Thus we can apply Exercise 5.26 to the sum $\sum_{k=1}^m (A_{J_k} A_{J_k}^\top - \mathbb{E} A_{J_k} A_{J_k}^\top)$, and that with probability at least

$1 - e^{-u}$,

$$\left\|B^\top B - \frac{m}{N} A^\top A\right\| \le C' \left(\sqrt{\frac{n(\log n + u)}{m}} + \frac{n(\log n + u)}{m}\right) \frac{m}{N} \left\|A^\top A\right\|$$

Choosing $u = \log 10$ and taking the constant C in $m \ge Cn \log n$ sufficiently large, we get

$$\left\|B^\top B - \frac{m}{N} A^\top A\right\| \le 0.1 \frac{m}{N} \left\|A^\top A\right\| = 0.1 \frac{m}{N} s_1(A)^2$$

with probability at least 0.9. Equivalently,

$$\left\|\frac{N}{m} B^\top B - A^\top A\right\| \le 0.1\, s_1(A)^2.$$

By Weyl's inequality,

$$\max_{i=1,\dots,n} \left|\lambda_i(A^\top A) - \lambda_i\left(\frac{N}{m} B^\top B\right)\right| \le 0.1\, s_1(A)^2.$$

Since $\lambda_i(A^\top A) = s_i(A)^2$ and $\lambda_i(\frac{N}{m} B^\top B) = \frac{N}{m} s_i(B)^2$, this gives the desired bound.

6 Quadratic Forms, Symmetrization and Contraction

6.2 (L^p and subgaussian decoupling) Let $(a_{ij})_{i,j=1}^n$ be real numbers. Let $X_1, \dots, X_n$ be independent, mean-zero random variables, and (X_i') be an independent copy of (X_i).

(a) For any $p \in [1, \infty)$, show that

$$\left\|\sum_{i,j:\, i \ne j} a_{ij} X_i X_j\right\|_{L^p} \le 4 \left\|\sum_{i,j} a_{ij} X_i X_j'\right\|_{L^p}.$$

(b) Show that

$$\left\|\sum_{i,j:\, i \ne j} a_{ij} X_i X_j\right\|_{\psi_2} \le 4 \left\|\sum_{i,j} a_{ij} X_i X_j'\right\|_{\psi_2}.$$

Solution: (a) The result follows immediately using Exercise 6.1 with $F(x) = |x|^p$.

(b) We use Exercise 6.1 with $F(x) = \exp(x^2/c)$, where c is chosen as $16 \left\| \sum_{i,j} a_{ij} X_i X'_j \right\|_{\psi_2}$:

$$\mathbb{E} \exp\left(\frac{\left| \sum_{i,j:\, i \neq j} a_{ij} X_i X_j \right|^2}{16 \left\| \sum_{i,j} a_{ij} X_i X'_j \right\|_{\psi_2}^2} \right) \leq \mathbb{E} \exp\left(\frac{\left| \sum_{i,j} a_{ij} X_i X'_j \right|^2}{\left\| \sum_{i,j} a_{ij} X_i X'_j \right\|_{\psi_2}^2} \right) \leq 2.$$

This implies the desired result.

6.4 (Decoupling the norm of a random submatrix) Let A be an $n \times n$ diagonal-free matrix. Pick a random subset $J \subset \{1, \ldots, n\}$ by including each index independently with probability $p \in (0, 1)$. Let J' be an independent copy of J. Show that

$$\mathbb{E} \|A_{J \times J}\| \leq 4 \mathbb{E} \|A_{J \times J'}\|.$$

Here $A_{I \times J}$ denotes the submatrix of A with rows from I and columns from J.

Solution: Let $X_1, \ldots, X_n$ be a sequence of independent Bernoulli(p) variables, and $X'_1, \ldots, X'_n$ be an independent copy of $X_1, \ldots, X_n$. Define A_{ij} as the matrix with the same size as A but with all entries zero except for the (i, j)-th entry. Then we have $A_{J \times J} = \sum_{i,j} A_{ij} X_i X_j$, and $A_{J \times J'} = \sum_{i,j} A_{ij} X_i X'_j$.

Following the same argument as in the proof of Theorem 6.1.1, consider a sequence of independent Bernoulli(p) variables $\delta_1, \ldots, \delta_n$ and let $I = \{i \colon X_i = 1\}$ be a random subset of $\{1, \ldots, n\}$, we have

$$\sum_{i,j} A_{ij} X_i X_j = 4 \mathbb{E}_I \sum_{i \in I, j \in I^c} A_{ij} X_i X_j.$$

By Jensen's inequality and Fubini's theorem, we have

$$\mathbb{E}_X \left\| \sum_{i,j} A_{ij} X_i X_j \right\| \leq 4 \mathbb{E}_I \mathbb{E}_X \left\| \sum_{i \in I, j \in I^c} A_{ij} X_i X_j \right\|.$$

Therefore, there exists a subset I_0 such that

$$\mathbb{E}_X \left\| \sum_{i,j} A_{ij} X_i X_j \right\| \le 4\mathbb{E}_X \left\| \sum_{i \in I_0, j \in I_0^c} A_{ij} X_i X_j \right\| = 4\mathbb{E}_X \left\| \sum_{i \in I_0, j \in I_0^c} A_{ij} X_i X_j' \right\|.$$

By Exercise 4.2 (d), the operator norm of any submatrix is bounded by the operator norm of the original matrix. Thus

$$4\mathbb{E}_X \left\| \sum_{i \in I_0, j \in I_0^c} A_{ij} X_i X_j' \right\| \le 4\mathbb{E}_X \left\| \sum_{i,j} A_{ij} X_i X_j' \right\|.$$

This completes the proof.

6.5 (No deviation bound from the mean) It is tempting to conjecture that the first-order term in Proposition 6.2.1 should be the mean of $\|X\|$, which is approximately $\sqrt{n}$, instead of $CK\sqrt{n}$. But that is false! Give an example showing that the bound

$$\mathbb{P}\left\{ \|X\|_2 \ge C\left(\sqrt{n} + Kt\right) \right\} \le e^{-t^2} \quad \text{for all } t \ge 0$$

does not always hold.

Solution: First, Proposition 6.2.1 itself does not assume that X is isotropic, so $\mathbb{E}\|X\|_2$ need not be approximately $\sqrt{n}$.

Even after adding the isotropicity assumption, the proposed bound is still false. Let $g \sim \mathcal{N}(0, I_n)$, and let η be independent of g with $\mathbb{P}\{\eta = R\} = 1/(2R^2)$ and $\mathbb{P}\{\eta = a\} = 1 - 1/(2R^2)$, where a is chosen so that $\mathbb{E}\eta^2 = 1$. Set $X = \eta g$, then X is mean-zero and isotropic. Also, $\|X\|_{\psi_2} \le C_0 R$.

Assume the claimed inequality held with an absolute constant C. Take $t = c\sqrt{n}$, with $c > 0$ small enough such that $C(\sqrt{n} + \|X\|_{\psi_2} t) \le R\sqrt{n}/2$ for all sufficiently large R. On the event $\{\eta = R,\ \|g\|_2 \ge \sqrt{n}/2\}$, we have $\|X\|_2 \ge R\sqrt{n}/2$. Hence the left hand side is at least

$$\mathbb{P}\{\eta = R\}\mathbb{P}\{\|g\|_2 \ge \sqrt{n}/2\} \ge \frac{c'}{R^2},$$

whereas the claimed right hand side equals $e^{-t^2} = e^{-cn}$. Choosing $n \gg \log R$ gives $e^{-cn} \ll R^{-2}$, which is a contradiction.

6.6 (Norms of random matrices with subgaussian columns)

(a) Extend the maximal inequality (Exercise 3.13) to subgaussian random vectors, even without independent entries.

(b) Extend the bounds on the $1 \to \infty$ and $1 \to 2$ norms bound (Exercise 4.44(a),(b)) to random matrices with independent subgaussian columns, even if entries are not independent.

Solution: (a) Let $X_1, \dots, X_N$ be mean-zero, subgaussian random vectors in $\mathbb{R}^n$ with $\max_i \|X_i\|_{\psi_2} \leq K$. We want to show that

$$\mathbb{E} \max_i \|X_i\|_2 \leq CK(\sqrt{n} + \sqrt{\log N}).$$

By Proposition 6.2.1, we have

$$\mathbb{P}(\|X_i\|_2 \geq CK(\sqrt{n} + t)) \leq e^{-t^2}.$$

By the union bound, we have

$$\mathbb{P}(\max_i \|X_i\|_2 \geq CK(\sqrt{n} + t)) \leq Ne^{-t^2}.$$

Integrating the tail bound gives

$$\begin{aligned}
\mathbb{E} \max_i \|X_i\|_2 &= \int_0^\infty \mathbb{P}(\max_i \|X_i\|_2 \geq t) dt \\
&\leq CK\sqrt{n} + CK \int_0^\infty \min\{1, Ne^{-t^2}\} \, dt \\
&\leq CK\sqrt{n} + CK \left(\sqrt{\log N} + N \int_{\sqrt{\log N}}^\infty e^{-t^2} \, dt \right) \\
&\leq CK\sqrt{n} + CK(\sqrt{\log N} + 1) \\
&\leq CK\sqrt{n} + CK\sqrt{\log N}.
\end{aligned}$$

This is the analogue of Exercise 3.13 in terms of the subgaussian norm of the whole vector. The first term becomes $CK\sqrt{n}$, as Proposition 6.2.1 and Exercise 6.5 indicate.

(b) Let A be an $m \times n$ random matrix whose columns $A_{:1}, \dots, A_{:n}$ are independent, mean-zero, subgaussian random vectors in $\mathbb{R}^m$, and let $K = \max_{j \leq n} \|A_{:j}\|_{\psi_2}$. We claim that

$$\mathbb{E}\|A\|_{1\to\infty} \leq CK(\sqrt{\log m} + \sqrt{\log n}), \qquad \mathbb{E}\|A\|_{1\to 2} \leq CK(\sqrt{m} + \sqrt{\log n}).$$

For the first bound, Exercise 4.9 gives $\|A\|_{1\to\infty} = \max_{i\le m, j\le n} |A_{ij}|$. Since $\|A_{ij}\|_{\psi_2} \le \|A_{:j}\|_{\psi_2} \le K$, Proposition 2.7.6 yields

$$\mathbb{E}\|A\|_{1\to\infty} = \mathbb{E} \max_{i\le m, j\le n} |A_{ij}| \le CK\sqrt{\log(mn)} \le CK(\sqrt{\log m} + \sqrt{\log n}).$$

For the second bound, again by Exercise 4.9, $\|A\|_{1\to 2} = \max_{j\le n} \|A_{:j}\|_2$. Applying part (a) to the n subgaussian columns in $\mathbb{R}^m$, we get

$$\mathbb{E}\|A\|_{1\to 2} = \mathbb{E} \max_{j\le n} \|A_{:j}\|_2 \le CK(\sqrt{m} + \sqrt{\log n}).$$

6.7 (Gaussian Hanson-Wright) Give an alternative proof of Hanson-Wright inequality for normal distributions, without separating the diagonal part or decoupling.

Solution: Let $g \sim N(0, I_n)$, and let A be a fixed $n\times n$ matrix. If A is not symmetric, we consider the symmetric version of A: $B = \frac{A+A^\top}{2}$, then $g^\top Ag = g^\top Bg$. Moreover, $\|B\| \le \|A\|$, $\|B\|_F \le \|A\|_F$. Thus it suffices to prove the result for symmetric A.

Assume $A = A^\top$. By the spectral decomposition, $A = U^\top \Lambda U$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$. By the rotation invariance of the Gaussian distribution, $Ug \sim g$. Hence

$$g^\top Ag - \mathbb{E}g^\top Ag \overset{d}{=} \sum_{i=1}^n \lambda_i (g_i^2 - 1).$$

The random variables $g_i^2 - 1$ are independent, mean-zero, subexponential random variables with $\left\|g_i^2 - 1\right\|_{\psi_1} \le C$. Thus Bernstein's inequality gives

$$\mathbb{P}\left\{\sum_{i=1}^n \lambda_i(g_i^2 - 1) \ge t\right\} \le \exp\left[-c\min\left(\frac{t^2}{\|\lambda\|_2^2}, \frac{t}{\|\lambda\|_\infty}\right)\right],$$

here we define $\lambda = (\lambda_1, \dots, \lambda_n)$. Applying the same argument to $-A$ gives the lower tail. Since $\|\lambda\|_2 = \|A\|_F$ and $\|\lambda\|_\infty = \|A\|$, we conclude

$$\mathbb{P}\left\{\left|g^\top Ag - \mathbb{E}g^\top Ag\right| \ge t\right\} \le 2\exp\left[-c\min\left(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|}\right)\right].$$

6.8 (Higher-dimensional Hanson-Wright) Let $A = (a_{ij})$ be an $n \times n$ matrix. Let $X_1, \dots, X_n$ be independent, mean-zero, subgaussian random vectors in $\mathbb{R}^d$. Prove that for every $t \geq 0$, we have

$$\mathbb{P}\left\{\left|\sum_{i,j:\, i\neq j} a_{ij}\langle X_i, X_j\rangle\right| \geq t\right\} \leq 2\exp\left[-c\min\left(\frac{t^2}{K^4 d\|A\|_F^2}, \frac{t}{K^2\|A\|}\right)\right],$$

where $K = \max_i \|X_i\|_{\psi_2}$.

Solution: Denote $S = \sum_{i,j:\, i\neq j} a_{ij}\langle X_i, X_j\rangle$, and let $X_1', \dots, X_n'$ be an independent copy of $X_1, \dots, X_n$. As in Exercise 6.2, the decoupling estimate with $F(x) = e^{\lambda x}$ gives

$$\mathbb{E}e^{\lambda S} \leq \mathbb{E}\exp\left(4\lambda \sum_{i,j} a_{ij}\langle X_i, X_j'\rangle\right), \qquad \lambda \geq 0.$$

Set $\mathbf{X} = (X_1, \dots, X_n) \in \mathbb{R}^{nd}$, $\mathbf{X'} = (X_1', \dots, X_n')$, and $B = A \otimes I_d$. Then $\sum_{i,j} a_{ij}\langle X_i, X_j'\rangle = \mathbf{X}^T B \mathbf{X'}$. Moreover $\|\mathbf{X}\|_{\psi_2} \leq CK$, since for $u = (u_1, \dots, u_n) \in S^{nd-1}$, the sum $\sum_i \langle X_i, u_i\rangle$ has subgaussian norm at most $CK(\sum_i \|u_i\|_2^2)^{1/2} = CK$.

Applying Gaussian replacement (Lemma 6.2.3) to $\mathbf{X}^T B \mathbf{X'}$, and then the Gaussian MGF bound (Lemma 6.2.4), we get

$$\mathbb{E}e^{\lambda S} \leq \mathbb{E}e^{C_0 K^2 \lambda g^T B g'} \leq \exp(C_1 K^4 \lambda^2 \|B\|_F^2)$$

whenever $\lambda \leq c_0/(K^2\|B\|)$. Since $\|B\|_F = \sqrt{d}\,\|A\|_F$ and $\|B\| = \|A\|$, Markov's inequality gives

$$\mathbb{P}\{S \geq t\} \leq \exp\left(-\lambda t + C_1 K^4 d\lambda^2 \|A\|_F^2\right), \qquad 0 \leq \lambda \leq \frac{c_0}{K^2\|A\|}.$$

Optimizing over λ gives

$$\mathbb{P}\{S \geq t\} \leq \exp\left[-c\min\left(\frac{t^2}{K^4 d\|A\|_F^2}, \frac{t}{K^2\|A\|}\right)\right].$$

Applying the same argument to $-S$ completes the proof.

6.9 (MGF of the squared norm) Let B be an $m \times n$ matrix, and let X be a mean-zero subgaussian random vector in $\mathbb{R}^n$ with $\|X\|_{\psi_2} \leq K$. Show that

$$\mathbb{E}\exp\left(\lambda^2\|BX\|_2^2\right) \leq \exp\left(CK^2\lambda^2\|B\|_F^2\right)$$

whenever $|\lambda| \le c/(K\,\|B\|)$.

Solution: Let $g \sim N(0, I_m)$ be independent of X. Conditional on X, we have

$$\mathbb{E}_g \exp\left(\sqrt{2}\lambda\langle g, BX\rangle\right) = \exp\left(\lambda^2 \|BX\|_2^2\right).$$

Therefore, by Fubini and the subgaussian MGF bound,

$$\mathbb{E}\exp\left(\lambda^2 \|BX\|_2^2\right) = \mathbb{E}_g\mathbb{E}_X \exp\left(\sqrt{2}\lambda\langle X, B^T g\rangle\right) \le \mathbb{E}_g \exp\left(CK^2\lambda^2 \left\|B^T g\right\|_2^2\right).$$

Let $s_1, \dots, s_r$ be the singular values of B. By rotation invariance of g, $\left\|B^T g\right\|_2^2 \overset{d}{=} \sum_{i=1}^r s_i^2 g_i^2$, so

$$\mathbb{E}_g \exp\left(CK^2\lambda^2 \left\|B^T g\right\|_2^2\right) \le \prod_{i=1}^r (1 - CK^2\lambda^2 s_i^2)^{-1/2},$$

If $|\lambda| \le c/(K\,\|B\|)$, the factors are well-defined and $(1-x)^{-1/2} \le e^x$ for all $x = CK^2\lambda^2 s_i^2$ after decreasing c. Hence

$$\mathbb{E}\exp\left(\lambda^2 \|BX\|_2^2\right) \le \exp\left(CK^2\lambda^2 \sum_i s_i^2\right) = \exp\left(CK^2\lambda^2 \|B\|_F^2\right).$$

6.10 (The norm of an anisotropic random vector) Let us extend Proposition 6.2.1 for anisotropic random vectors. Let B be an $m \times n$ matrix, and let X be a mean-zero sub-gaussian random vector in $\mathbb{R}^n$ with $\|X\|_{\psi_2} \le K$. For every $t \ge 0$, show that

$$\mathbb{P}\{\|BX\|_2 \ge CK\,(\|B\|_F + t\|B\|)\} \le e^{-t^2}.$$

Solution: Let C, c be the constants in the MGF bound in Exercise 6.9 and choose C' such that $(C')^2 \ge C + c^{-2}$. We have

$$\begin{aligned}
\mathbb{P}\{\|BX\|_2 \ge C'K(\|B\|_F + t\|B\|)\} &= \mathbb{P}\{\|BX\|_2^2 \ge (C')^2K^2(\|B\|_F + t\|B\|)^2\} \\
&\le \mathbb{E}\exp\left(\lambda^2 \|BX\|_2^2\right) \exp\left(-(C')^2K^2\lambda^2(\|B\|_F + t\|B\|)^2\right) \\
&\le \exp\left(CK^2\lambda^2 \|B\|_F^2\right) \exp\left(-(C')^2K^2\lambda^2(\|B\|_F + t\|B\|)^2\right) \\
&\le \exp\left(-(C')^2K^2\lambda^2 t^2 \|B\|^2\right)
\end{aligned}$$

where $|\lambda| \le c/(K\,\|B\|)$. Now let $|\lambda| = c/(K\,\|B\|)$, we get

$$\mathbb{P}\{\|BX\|_2 \ge C'K(\|B\|_F + t\|B\|)\} \le \exp(-(cC')^2 t^2) \le \exp(-t^2).$$

This completes the proof after adjusting the constant C.

6.11 (Hanson-Wright for subgaussian vectors) Here is a useful version of Hanson-Wright inequality (Theorem 6.2.2) without assuming independence of entries. Let A be a symmetric positive-semidefinite $n \times n$ matrix, and let X be a mean-zero subgaussian random vector in $\mathbb{R}^n$ with $\|X\|_{\psi_2} \le K$. For any $s \ge 0$, show that

$$\mathbb{P}\left\{X^T AX \ge CK^2\,(\operatorname{tr} A + s\|A\|)\right\} \le e^{-s}.$$

(To compare with Theorem 6.2.2, set $t = CK^2 s\|A\|$ and note that $\operatorname{tr} A = \mathbb{E}X^T AX$ for isotropic X.) Explain why there can be only one-sided bound here, even if X is isotropic.

Solution: We can write $X^T AX = \|A^{1/2}X\|_2^2$. By Exercise 6.10, we have

$$\mathbb{P}\{\|A^{1/2}X\|_2 \ge CK(\|A^{1/2}\|_F + t\|A^{1/2}\|)\} \le e^{-t^2}.$$

Since $\|A^{1/2}\|_F^2 = \operatorname{tr} A$ and $\|A^{1/2}\|^2 = \|A\|$, the desired bound follows by setting $t = \sqrt{s}$. The lower tail bound does not hold in general, even if X is isotropic. Let $B \sim$ Bernoulli(1/2), let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ have independent Rademacher entries, and assume B and ε are independent. Define $X = \sqrt{2}\,B\varepsilon$. Then X is mean zero and isotropic. Also, X is subgaussian with $\|X\|_{\psi_2} \le C$.

Now take $A = I_n$. Then

$$X^T AX = \|X\|_2^2 = \begin{cases} 0, & \text{with probability } 1/2, \\ 2n, & \text{with probability } 1/2. \end{cases}$$

The probability $\mathbb{P}(X^\top AX = 0)$ does not decay with n.

6.12 (Mean estimation) Let's revisit the mean estimation problem (Section 2.4), where we sample N i.i.d. points $X_1, \dots, X_N$ from some unknown distribution in $\mathbb{R}^n$ with mean μ and covariance

Σ, and we want to estimate μ. Assume the distribution is subgaussian:

$$\|\langle X_i - \mu, u\rangle\|_{\psi_2} \le K\|\langle X_i - \mu, u\rangle\|_{L^2} \quad \text{for all } u \in \mathbb{R}^n.$$

For any $\alpha \in (0,1)$, show the sample mean $\mu_N := \frac{1}{N}\sum_{i=1}^N X_i$ satisfies

$$\|\mu_N - \mu\|_2 \le CK\sqrt{\frac{\operatorname{tr}\Sigma}{N}} + CK\sqrt{\frac{\|\Sigma\|\log(1/\alpha)}{N}}$$

with probability at least $1-\alpha$.

Solution: WLOG, we assume $\mu = 0$ and Σ is invertible. (If Σ is not invertible, we can work in the subspace spanned by the support of the distribution, where Σ' is invertible.) Setting $X_i = \Sigma^{1/2}Z_i$, the assumption $\|\langle X_i, u\rangle\|_{\psi_2} \le K\|\langle X_i, u\rangle\|_{L_2}$ for all $u \in \mathbb{R}^n$ implies that Z_i are subgaussian with $\|Z_i\|_{\psi_2} \le K$. Let $\hat{\nu}_n = \frac{1}{N}\sum_{i=1}^N Z_i$, then $\|\hat{\nu}_n\|_{\psi_2} \le K/\sqrt{N}$. By Exercise 6.10, we have

$$\mathbb{P}\left\{\|\Sigma^{1/2}\hat{\nu}_n\|_2 \ge C\frac{K}{\sqrt{N}}\left(\left\|\Sigma^{1/2}\right\|_F + t\left\|\Sigma^{1/2}\right\|\right)\right\} \le e^{-t^2}.$$

The result follows by setting $t = \sqrt{\log(1/\alpha)}$ and noting that $\|\Sigma^{1/2}\|_F^2 = \operatorname{tr}\Sigma$ and $\|\Sigma^{1/2}\|^2 = \|\Sigma\|$.

6.13 (Anisotropic concentration of the norm) Let's extend the concentration of norm (Theorem 3.1.1) to anisotropic distributions. Let B be an $m \times n$ matrix, and let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with independent, mean-zero, unit variance, subgaussian coordinates. Show that

$$\|\|BX\|_2 - \|B\|_F\|_{\psi_2} \le CK^2\|B\|,$$

where $K = \max_i \|X_i\|_{\psi_2}$.

Solution: Note that $\mathbb{E}\|BX\|_2^2 = \|B\|_F^2$. Applying Hanson-Wright inequality (Theorem 6.2.2) to $B^\top B$, we get for every $u \ge 0$,

$$\mathbb{P}\left\{\left|\|BX\|_2^2 - \|B\|_F^2\right| \ge u\|B\|_F^2\right\} \le 2\exp\left[-c\min\left(\frac{u^2\|B\|_F^4}{K^4\|B^\top B\|_F^2}, \frac{u\|B\|_F^2}{K^2\|B^\top B\|}\right)\right].$$

Now $\|B^\top B\| = \|B\|^2$, and by Exercise 4.4(c),

$$\|B^\top B\|_F \le \|B^\top\|\|B\|_F = \|B\|\|B\|_F.$$

Also, K is bounded below by an absolute constant because $1 = \|X_i\|_{L^2} \le C\|X_i\|_{\psi_2} \le CK$. Thus

$$\mathbb{P}\left\{\left|\frac{\|BX\|_2^2}{\|B\|_F^2} - 1\right| \ge u\right\} \le 2\exp\left[-c\frac{\|B\|_F^2}{K^4\|B\|^2}\min(u^2, u)\right].$$

We now convert this squared-norm bound to a norm bound as in the proof of Theorem 3.1.1. For all $z, \delta \ge 0$,

$$|z-1| \ge \delta \quad \text{implies} \quad |z^2 - 1| \ge \max(\delta, \delta^2).$$

Applying this with $z = \|BX\|_2/\|B\|_F$ and $u = \max(\delta, \delta^2)$, we obtain

$$\mathbb{P}\left\{\left|\frac{\|BX\|_2}{\|B\|_F} - 1\right| \ge \delta\right\} \le 2\exp\left[-c\frac{\|B\|_F^2\delta^2}{K^4\|B\|^2}\right], \qquad \delta \ge 0.$$

Changing variables $t = \delta\|B\|_F$, this becomes

$$\mathbb{P}\left\{|\|BX\|_2 - \|B\|_F| \ge t\right\} \le 2\exp\left[-c\frac{t^2}{K^4\|B\|^2}\right], \qquad t \ge 0.$$

By Proposition 2.6.6, this tail bound is equivalent to

$$\big\|\|BX\|_2 - \|B\|_F\big\|_{\psi_2} \le CK^2\|B\|.$$

6.14 (Distance from a random vector to a subspace) Let E be a subspace of $\mathbb{R}^n$ of dimension d. Let $X = (X_1, \dots, X_n) \in \mathbb{R}^n$ be a random vector with independent, mean-zero, unit variance, subgaussian coordinates.

(a) Check that

$$\big(\mathbb{E}\,\mathrm{dist}(X, E)^2\big)^{1/2} = \sqrt{n-d}.$$

(b) Deduce from Exercise 6.13 that for any $t \ge 0$, the distance nicely concentrates:

$$\mathbb{P}\left\{\left|\mathrm{dist}(X, E) - \sqrt{n-d}\right| > t\right\} \le 2\exp\left(-\frac{ct^2}{K^4}\right),$$

where $K = \max_i \|X_i\|_{\psi_2}$.

Solution: (a) WLOG, we can assume E is the subspace spanned by the first d standard basis vectors. Then $\mathrm{dist}(X, E)^2 = \sum_{i=d+1}^n X_i^2$. Since X_i are independent, mean-zero, unit variance, we

have

$$\mathbb{E}\,\mathrm{dist}(X,E)^2 = \sum_{i=d+1}^{n} \mathbb{E}X_i^2 = n-d.$$

(b) Like in (a), we assume E is the subspace spanned by the first d standard basis vectors. Then $\mathrm{dist}(X,E) = \|BX\|_2$, where B is the $(n-d)\times n$ matrix that projects onto the last $n-d$ coordinates. By Exercise 6.13, we have $\big\|\|BX\|_2 - \|B\|_F\big\|_{\psi_2} \le CK^2\|B\|$. Since $\|B\|_F = \sqrt{n-d}$ and $\|B\| = 1$, we have $\big\|\mathrm{dist}(X,E) - \sqrt{n-d}\big\|_{\psi_2} \le CK^2$. The conclusion follows by applying Proposition 2.6.6.

6.15 (Randomly cutting a graph) Take any graph with E edges and randomly split its vertices into two groups (as in Section 3.6.3). As we saw in Proposition 3.6.3, the expected "cut" (the number of crossing edges) is $E/2$. Now show that it is tightly concentrated around that value:

$$\mathbb{P}\left\{\left|\mathrm{cut} - \frac{E}{2}\right| \ge s\sqrt{E}\right\} \le 2e^{-cs} \quad \text{for any } s \ge 1.$$

Solution: Let $X_1,\dots,X_n$ be independent Rademacher random variables, where n is the number of vertices in the graph. Let $A = (a_{ij})$ be the adjacency matrix of the graph. Then the number of crossing edges is given by $\mathrm{cut} = \frac{1}{4}\sum_{i,j} a_{ij}(1 - X_iX_j)$. And we have

$$\left|\mathrm{cut} - \frac{E}{2}\right| = \left|X^\top\left(\frac{A}{4}\right)X\right|, \quad \mathbb{E}X^\top\left(\frac{A}{4}\right)X = 0$$

By Hanson-Wright inequality, we have

$$\mathbb{P}\left\{\left|X^\top\left(\frac{A}{4}\right)X\right| \ge s\sqrt{E}\right\} \le 2\exp\left[-c\min\left(\frac{s^2E}{\|A/4\|_F^2}, \frac{s\sqrt{E}}{\|A/4\|}\right)\right].$$

Since $\|A/4\|_F^2 = \sum_{i,j} a_{ij}^2/16 = E/16$ and $\|A/4\| = \|A\|/4 \le \sqrt{E}/4$, we have

$$\mathbb{P}\left\{\left|X^\top\left(\frac{A}{4}\right)X\right| \ge s\sqrt{E}\right\} \le 2\exp\left[-c\min\left(16s^2, 4s\right)\right] \le 2e^{-cs} \quad \text{for any } s > 1.$$

6.18 (Norm of a shifted random vector) Let X be a symmetric random vector (meaning X and $-X$ have the same distribution) taking values in some normed space, and v be a fixed vector in that space. Show that

$$\mathbb{E}\|X+v\| \lesssim \mathbb{E}\|X\| + \|v\|,$$

where the notation $\lesssim$ hides positive absolute constant factors.

Solution: It is trivial to see that $\mathbb{E}\|X+v\| \le \mathbb{E}\|X\| + \|v\|$ by the triangle inequality. To show the other direction, we note

$$\mathbb{E}\,\|X+v\| + \mathbb{E}\,\|X-v\| \ge 2\mathbb{E}\,\|X\|\,, \quad \mathbb{E}\,\|-X+v\| + \mathbb{E}\,\|X+v\| \ge 2\,\|v\|\,.$$

Since X is symmetric, we have $\mathbb{E}\,\|X+v\| = \mathbb{E}\,\|-X+v\| = \mathbb{E}\,\|X-v\|$, so

$$4\mathbb{E}\,\|X+v\| \ge 2\mathbb{E}\,\|X\| + 2\,\|v\|\,,$$

which implies $\mathbb{E}\,\|X+v\| \ge \frac{1}{2}\,(\mathbb{E}\,\|X\| + \|v\|)$. We get the desired result.

6.19 (Symmetrization without zero mean)

(a) Prove the following generalization of Symmetrization Lemma 6.3.2 for random vectors X_i that do not necessarily have zero means:

$$\mathbb{E}\left\|\sum_{i=1}^N X_i - \sum_{i=1}^N \mathbb{E}X_i\right\| \le 2\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i X_i\right\|.$$

(b) Argue that there can not be any non-trivial reverse inequality.

Solution: (a) Let $X_1', \ldots, X_N'$ be an independent copy of $X_1, \ldots, X_N$. Then by Jensen's inequality, we have

$$\mathbb{E}\left\|\sum_{i=1}^N (X_i - \mathbb{E}X_i)\right\| \le \mathbb{E}\left\|\sum_{i=1}^N (X_i - \mathbb{E}X_i) - \sum_{i=1}^N (X_i' - \mathbb{E}X_i')\right\| = \mathbb{E}\left\|\sum_{i=1}^N (X_i - X_i')\right\|.$$

Since $(X_i - X_i')$ are symmetric random vectors, they have the same distribution as $(\varepsilon_i(X_i - X_i'))$. The result follows by the triangle inequality:

$$\mathbb{E}\left\|\sum_{i=1}^N (X_i - X_i')\right\| = \mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i(X_i - X_i')\right\| \le \mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i X_i\right\| + \mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i X_i'\right\| = 2\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i X_i\right\|.$$

(b) One can see the reverse inequality does not hold in general by simply considering the case where X_i are deterministic vectors.

6.22 (Self-normalized sums) Let $X_1, \ldots, X_N$ be independent symmetric random variables. Show that, for any $t > 0$,

$$\mathbb{P}\left\{\left|\sum_{i=1}^N X_i\right| \geq t\left(\sum_{i=1}^N X_i^2\right)^{1/2}\right\} \leq 2e^{-t^2/2}$$

This is very general –no subgaussian or moment assumptions needed!

Solution: Since $X_1, \ldots, X_N$ are independent symmetric random variables, $\sum_{i=1}^N X_i$ has the same distribution as $\sum_{i=1}^N \varepsilon_i X_i$, where ε_i are independent Rademacher random variables. Therefore, we can write

$$\mathbb{P}\left\{\left|\sum_{i=1}^N X_i\right| \geq t\left(\sum_{i=1}^N X_i^2\right)^{1/2}\right\} = \mathbb{P}\left\{\left|\sum_{i=1}^N \varepsilon_i X_i\right| \geq t\left(\sum_{i=1}^N X_i^2\right)^{1/2}\right\}.$$

Conditioning on $X_1, \ldots, X_N$, we can apply Hoeffding's inequality to the right-hand side:

$$\mathbb{P}\left\{\left|\sum_{i=1}^N \varepsilon_i X_i\right| \geq t\left(\sum_{i=1}^N X_i^2\right)^{1/2}\right\} \leq 2\exp\left(-\frac{t^2\sum_{i=1}^N X_i^2}{2\sum_{i=1}^N X_i^2}\right) = 2e^{-t^2/2}.$$

The result follows by taking expectation over $X_1, \ldots, X_N$.

6.23 (Type p). Let $p \in [1, 2]$.

(a) Let $x_1, \ldots, x_N \in \mathbb{R}^n$ be any (fixed) vectors. Check that

$$\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p^p \leq \sum_{i=1}^N \|x_i\|_p^p.$$

(b) Generalize part (a) as follows. Let $X_1, \ldots, X_N$ be mean zero, independent random vectors in $\mathbb{R}^n$. Show that

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\|_p^p \leq 2^p \sum_{i=1}^N \mathbb{E}\|X_i\|_p^p.$$

(c) Show by example that the results in parts (a)–(b) fail for each $p > 2$.

Solution: (a) Recall that the L^p norm of random variables is increasing in p while the ℓ^p norm

of vectors is decreasing in p. Let x_{ik} be the k-th coordinate of x_i, for any $p \in [1,2]$,

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p^p &= \sum_{k=1}^n \mathbb{E}\left|\sum_{i=1}^N \varepsilon_i x_{ik}\right|^p \\
&= \sum_{k=1}^n \left\|\sum_{i=1}^N \varepsilon_i x_{ik}\right\|_{L_p}^p \\
&\le \sum_{k=1}^n \left\|\sum_{i=1}^N \varepsilon_i x_{ik}\right\|_{L_2}^p \\
&\le \sum_{k=1}^n \left(\sum_{i=1}^N |x_{ik}|^2\right)^{p/2} \\
&= \sum_{k=1}^n \|x_{\cdot k}\|_2^p \\
&\le \sum_{k=1}^n \|x_{\cdot k}\|_p^p \\
&= \sum_{i=1}^N \|x_i\|_p^p .
\end{aligned}$$

(b) For any $p \in [1,2]$, we can apply Exercise 6.20 with $F(x) = x^p$ to get

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\|_p^p \le 2^p \mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i X_i\right\|_p^p .$$

Conditioning on $X_1, \dots, X_N$ and applying part (a) gives

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\|_p^p \le 2^p \mathbb{E}\sum_{i=1}^N \|X_i\|_p^p = 2^p \sum_{i=1}^N \mathbb{E}\|X_i\|_p^p.$$

(c) Fix any $p > 2$, we show that both inequalities fail already in dimension $n = 1$. Let $x_1 = \cdots = x_N = 1 \in \mathbb{R}$. Since the L^p norm of random variables is increasing in p,

$$\mathbb{E}\left|\sum_{i=1}^N \varepsilon_i x_i\right|^p \ge \left(\mathbb{E}\left|\sum_{i=1}^N \varepsilon_i x_i\right|^2\right)^{p/2} = N^{p/2}.$$

On the other hand, $\sum_{i=1}^N |x_i|^p = N$. Since $p > 2$, choosing N sufficiently large gives $N^{p/2} > N$, so

part (a) fails. The same construction, with $X_i = \varepsilon_i$, gives

$$\mathbb{E}\left|\sum_{i=1}^N X_i\right|^p \geq N^{p/2}, \qquad 2^p \sum_{i=1}^N \mathbb{E}|X_i|^p = 2^p N.$$

Choosing N sufficiently large so that $N^{p/2} > 2^p N$, we see that part (b) fails as well.

6.24 (Type 2) Let $p \in [2, \infty)$. The constants C_1 and C_2 below are positive absolute constants.

(a) Let $x_1, \ldots, x_N \in \mathbb{R}^n$ be any (fixed) vectors. Check that

$$\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p^2 \leq C_1 p \sum_{i=1}^N \|x_i\|_p^2.$$

(b) Generalize part (a) as follows. Let $X_1, \ldots, X_N$ be mean zero, independent random vectors in $\mathbb{R}^n$. Show that

$$\mathbb{E}\left\|\sum_{i=1}^N X_i\right\|_p^2 \leq C_2 p \sum_{i=1}^N \mathbb{E}\|X_i\|_p^2.$$

(c) Show by example that the results in parts (a)–(b) fail for each $p < 2$.

Solution: (a) Let x_{ik} be the k-th coordinate of x_i, for any $p \in [2, \infty)$, we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p^2 &= \left\|\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p\right\|_{L_2}^2 \\
&\leq \left\|\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p\right\|_{L_p}^2 \\
&= \left(\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i x_i\right\|_p^p\right)^{2/p} \\
&= \left(\mathbb{E}\sum_{k=1}^n \left|\sum_{i=1}^N \varepsilon_i x_{ik}\right|^p\right)^{2/p} \\
&\leq \left(\sum_{k=1}^n \mathbb{E}\left|\sum_{i=1}^N \varepsilon_i x_{ik}\right|^p\right)^{2/p} \\
&= \left(\sum_{k=1}^n \left\|\sum_{i=1}^N \varepsilon_i x_{ik}\right\|_{L_p}^p\right)^{2/p}
\end{aligned}$$

$$
\begin{aligned}
&\leq \left(\sum_{k=1}^{n} C_0^p p^{p/2} \left(\sum_{i=1}^{N} x_{ik}^2 \right)^{p/2} \right)^{2/p} \\
&= C_0^2 p \left\| \sum_{i=1}^{N} x_i' \right\|_{p/2} \\
&\leq C_0^2 p \sum_{i=1}^{N} \left\| x_i' \right\|_{p/2} \\
&= C_1 p \sum_{i=1}^{N} \|x_i\|_p^2 .
\end{aligned}
$$

Here we used the monotonicity of the L^p norm of random variables, the Khintchine inequality, and the triangle inequality in $(\mathbb{R}^n, \|\cdot\|_{p/2})$. The vector x_i' has k-th coordinate x_{ik}^2, and $C_1 = C_0^2$.

(b) The proof is similar to that of Exercise 6.23(b). We can apply Exercise 6.20 with $F(x) = x^2$ to get

$$
\mathbb{E} \left\| \sum_{i=1}^{N} X_i \right\|_p^2 \leq 4\mathbb{E} \left\| \sum_{i=1}^{N} \varepsilon_i X_i \right\|_p^2 .
$$

Conditioning on $X_1, \ldots, X_N$ and applying part (a) gives

$$
\mathbb{E} \left\| \sum_{i=1}^{N} X_i \right\|_p^2 \leq 4C_1 p \sum_{i=1}^{N} \mathbb{E}\|X_i\|_p^2 = C_2 p \sum_{i=1}^{N} \mathbb{E}\|X_i\|_p^2,
$$

where $C_2 = 4C_1$.

(c) Fix $p < 2$. For part (a), take $n = N$ and $x_i = e_i$, the standard basis vectors in $\mathbb{R}^N$. Then

$$
\mathbb{E} \left\| \sum_{i=1}^{N} \varepsilon_i e_i \right\|_p^2 = N^{2/p}, \qquad \sum_{i=1}^{N} \|e_i\|_p^2 = N.
$$

Since $2/p > 1$, this contradicts the claimed bound for large N. For part (b), take $X_i = \varepsilon_i e_i$. Then the same computation gives

$$
\mathbb{E} \left\| \sum_{i=1}^{N} X_i \right\|_p^2 = N^{2/p}, \qquad \sum_{i=1}^{N} \mathbb{E}\|X_i\|_p^2 = N,
$$

so part (b) fails as well.

6.25 (Approximate ℓ_p-Carathéodory theorem) State and prove a version of the approximate Carathéodory theorem (Theorem 0.0.2) for the ℓ_p norm, for any $p \in [1, \infty)$.

Solution: We prove the following statement. Let $T \subset \mathbb{R}^n$ be contained in the unit ℓ_p-ball. Then, for every $x \in \operatorname{conv}(T)$ and every $k \in \mathbb{N}$, one can find points $x_1, \dots, x_k \in T$ such that

$$\left\| x - \frac{1}{k} \sum_{j=1}^{k} x_j \right\|_p \le \begin{cases} 4k^{-(1-1/p)}, & 1 \le p < 2, \\ C\sqrt{p/k}, & 2 \le p < \infty, \end{cases}$$

where C is an absolute constant.

The proof is similar to that of Theorem 0.0.2, but we need to use the type p property of ℓ_p spaces (Exercise 6.23 and Exercise 6.24) instead of the type 2 property of Hilbert spaces. Fix $x \in \operatorname{conv}(T)$ and write it as $x = \sum_{i=1}^m \lambda_i z_i$ with $z_i \in T$, $\lambda_i \ge 0$ and $\sum_{i=1}^m \lambda_i = 1$. Define a random vector Z that takes value z_i with probability λ_i. Then $\mathbb{E}Z = x$. Consider k independent copies $Z_1, \dots, Z_k$ of Z. If $p \in [1, 2)$, then by Exercise 6.23, we have

$$\mathbb{E} \left\| \sum_{j=1}^{k} (Z_j - x) \right\|_p^p \le 2^p \sum_{j=1}^{k} \mathbb{E} \|Z_j - x\|_p^p \le 2^p k \operatorname{diam}(T)^p \le 4^p k.$$

Therefore, there exists a realization of $Z_1, \dots, Z_k$ such that

$$\left\| \sum_{j=1}^{k} (Z_j - x) \right\|_p \le 4k^{1/p}.$$

For $p \in [2, \infty)$, by Exercise 6.24, we have

$$\mathbb{E} \left\| \sum_{j=1}^{k} (Z_j - x) \right\|_p^2 \le C_2 p \sum_{j=1}^{k} \mathbb{E} \|Z_j - x\|_p^2 \le C_2 pk \operatorname{diam}(T)^2 \le Cpk.$$

Therefore, there exists a realization of $Z_1, \dots, Z_k$ such that

$$\left\| \sum_{j=1}^{k} (Z_j - x) \right\|_p \le C\sqrt{pk}.$$

In either case, fix such a realization and call it $x_1, \dots, x_k$. Since each Z_j takes values in T, we have $x_j \in T$, and divide by k gives the desired approximation.

6.26 (Marcinkiewicz-Zygmund inequality) For independent, mean-zero random variables $X_1, \ldots, X_N$ and $p \in [2, \infty)$, show that

$$\left\| \sum_{i=1}^N X_i \right\|_{L^p}^2 \leq Cp \sum_{i=1}^N \|X_i\|_{L^p}^2.$$

Solution: By symmetrization (Exercise 6.20), we have

$$\left\| \sum_{i=1}^N X_i \right\|_{L^p}^2 \leq 4 \left\| \sum_{i=1}^N \varepsilon_i X_i \right\|_{L^p}^2.$$

Condition on $X_1, \ldots, X_N$ and apply Khintchine's inequality:

$$\mathbb{E}_\varepsilon \left| \sum_{i=1}^N \varepsilon_i X_i \right|^p \leq (C\sqrt{p})^p \left(\sum_{i=1}^N X_i^2 \right)^{p/2}.$$

Therefore,

$$\left\| \sum_{i=1}^N \varepsilon_i X_i \right\|_{L^p}^2 = \left(\mathbb{E}_X \mathbb{E}_\varepsilon \left| \sum_{i=1}^N \varepsilon_i X_i \right|^p \right)^{2/p} \leq Cp \left(\mathbb{E}_X \left(\sum_{i=1}^N X_i^2 \right)^{p/2} \right)^{2/p}.$$

By the triangle inequality in $L^{p/2}$,

$$\left(\mathbb{E}_X \left(\sum_{i=1}^N X_i^2 \right)^{p/2} \right)^{2/p} = \left\| \sum_{i=1}^N X_i^2 \right\|_{L^{p/2}} \leq \sum_{i=1}^N \|X_i^2\|_{L^{p/2}} = \sum_{i=1}^N \|X_i\|_{L^p}^2.$$

Combining the last two results with the symmetrization bound gives the desired result.

6.27 (Concentration of the norm under finite moments) Let's prove a version of Theorem 3.1.1 under weaker, finite-moment conditions. Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a random vector with independent coordinates X_i that satisfy $\mathbb{E}X_i^2 = 1$ and $\|X_i\|_{L^{2p}} \leq K$ for some $p \geq 2$ and $K \geq 0$. Show that

$$\big\| \|X\|_2 - \sqrt{n} \big\|_{L^p} \leq C\sqrt{p} K^2.$$

Solution: Recall that for any $z \geq 0$, $|z - 1| \geq \delta$ implies $|z^2 - 1| \geq \delta$. Therefore,

$$\big\| \|X\|_2 - \sqrt{n} \big\|_{L^p}^p = \big(\sqrt{n}\big)^p \, \mathbb{E} \left| \frac{\|X\|_2}{\sqrt{n}} - 1 \right|^p$$

$$
\begin{aligned}
&= \left(\sqrt{n}\right)^p \int_0^\infty p t^{p-1} \mathbb{P}\left\{ \left| \frac{\|X\|_2}{\sqrt{n}} - 1 \right| \geq t \right\} dt \\
&\leq \left(\sqrt{n}\right)^p \int_0^\infty p t^{p-1} \mathbb{P}\left\{ \left| \frac{\|X\|_2^2}{n} - 1 \right| \geq t \right\} dt \\
&= \left(\sqrt{n}\right)^p \mathbb{E}\left| \frac{\|X\|_2^2}{n} - 1 \right|^p \\
&= \left(\frac{1}{\sqrt{n}}\right)^p \mathbb{E}\left| \|X\|_2^2 - n \right|^p \\
&= \left(\frac{1}{\sqrt{n}}\right)^p \left\| \|X\|_2^2 - n \right\|_{L^p}^p .
\end{aligned}
$$

Now we can use the Marcinkiewicz-Zygmund inequality to get

$$
\begin{aligned}
\left\| \|X\|_2 - \sqrt{n} \right\|_{L^p}^2 &\leq \left(\frac{1}{n}\right) \left\| \|X\|_2^2 - n \right\|_{L^p}^2 \\
&\leq \left(\frac{1}{n}\right) \left\| \sum_{i=1}^n (X_i^2 - 1) \right\|_{L^p}^2 \\
&\leq \left(\frac{1}{n}\right) Cp \sum_{i=1}^n \left\| X_i^2 - 1 \right\|_{L^p}^2 \\
&\leq \left(\frac{1}{n}\right) Cp \sum_{i=1}^n 4 \left\| X_i^2 \right\|_{L^p}^2 \\
&\leq \left(\frac{1}{n}\right) Cp \sum_{i=1}^n 4 \left\| X_i \right\|_{L^{2p}}^4 \\
&\leq 4CpK^4.
\end{aligned}
$$

Taking square roots on both sides gives the desired result.

6.28 (Norm of rectangular random matrices) Extend Theorem 6.4.1 to non-symmetric, rectangular matrices. For an $m \times n$ random matrix A with independent, mean-zero entries, show that its expected operator norm is roughly the expected max Euclidean norm of its rows and columns, up to a logarithmic factor:

$$
\mathbb{E} \max_{i,j} \left(\|A_{i:}\|_2 , \|A_{:j}\|_2 \right) \leq \mathbb{E} \|A\| \leq C \sqrt{\log(m+n)}\, \mathbb{E} \max_{i,j} \left(\|A_{i:}\|_2 , \|A_{:j}\|_2 \right).
$$

Solution: Consider the symmetric dilation

$$B = \begin{pmatrix} 0 & A \\ A^\top & 0 \end{pmatrix}.$$

Then B is an $(m+n) \times (m+n)$ symmetric random matrix whose entries above the diagonal are independent and mean-zero. $\|B\| = \|A\|$ and the Euclidean norms of the rows of B are exactly the Euclidean norms of the rows and columns of A:

$$\max_{k \le m+n} \|B_{k:}\|_2 = \max \left\{ \max_{i \le m} \|A_{i:}\|_2 , \max_{j \le n} \|A_{:j}\|_2 \right\}.$$

Applying Theorem 6.4.1 to B therefore gives

$$\mathbb{E} \max_{i,j} \left(\|A_{i:}\|_2 , \|A_{:j}\|_2 \right) \le \mathbb{E} \|A\| \le C\sqrt{\log(m+n)}\, \mathbb{E} \max_{i,j} \left(\|A_{i:}\|_2 , \|A_{:j}\|_2 \right),$$

as required.

6.29 (The log factor is unremovable) Show that the logarithmic factor in Theorem 6.4.1 cannot be completely removed in general. Construct a random matrix A satisfying the assumptions of the theorem and for which

$$\mathbb{E} \|A\| \ge c \log^{1/4}(n) \cdot \mathbb{E} \max_i \|A_i\|_2 .$$

Solution: Consider a block diagonal matrix with n/k blocks, which are $k \times k$ symmetric random matrices with independent, Radermacher entries above and on the diagonal. Denote the matrix by A and the blocks by $B_1, \dots, B_{n/k}$. Suppose there exists i such that B_i is an all-ones matrix, then we have $\|A\| \ge \|B_i\| = k$. Since the different blocks are independent, then the probability that there exists i such that B_i is an all-ones matrix is $1 - \left(1 - \left(\frac{1}{2}\right)^{k(k+1)/2}\right)^{n/k}$. One the other hand, we always have $\max_i \|A_i\|_2 = \sqrt{k}$. Combining these observations and take gives

$$\mathbb{E} \|A\| \ge \left[1 - \left(1 - \left(\frac{1}{2} \right)^{k(k+1)/2} \right)^{n/k} \right] k \ge \left[1 - \exp\left(- \left(\frac{1}{2} \right)^{k^2} \frac{n}{k} \right) \right] k.$$

To get the desired result, it suffices to show that for some k we have

$$\left[1-\exp\left(-\left(\frac{1}{2}\right)^{k^2}\frac{n}{k}\right)\right]k \geq c\log^{1/4}(n)\sqrt{k},$$

which is equivalent to

$$\frac{\left[1-\exp\left(-\left(\frac{1}{2}\right)^{k^2}\frac{n}{k}\right)\right]\sqrt{k}}{\log^{1/4}(n)} \geq c.$$

Let $k = c'\log^{1/2}(n)$, then we have

$$\begin{aligned}\frac{\left[1-\exp\left(-\left(\frac{1}{2}\right)^{k^2}\frac{n}{k}\right)\right]\sqrt{k}}{\log^{1/4}(n)} &= \sqrt{c'}\left[1-\exp\left(-\left(\frac{1}{2}\right)^{(c')^2\log(n)}\frac{n}{c'\log^{1/2}(n)}\right)\right] \\ &= \sqrt{c'}\left[1-\exp\left(-\frac{1}{c'}\frac{n^{1-(c')^2\log(2)}}{\log^{1/2}(n)}\right)\right].\end{aligned}$$

Since $1-(c')^2\log(2) > 0$ for sufficiently small c', we have

$$\lim_{n\to\infty}\frac{\left[1-\exp\left(-\left(\frac{1}{2}\right)^{k^2}\frac{n}{k}\right)\right]\sqrt{k}}{\log^{1/4}(n)} = \sqrt{c'},$$

which yields the desired result.

6.30 (Norms of the rows of random matrices) Consider i.i.d. random variables $\delta_{ij} \sim \mathrm{Ber}(p)$, where $i, j = 1, \ldots, n$. Assuming that $pn \geq \log n$, show that

$$\mathbb{E}\max_{i\leq n}\sum_{j=1}^{n}(\delta_{ij}-p)^2 \leq Cpn.$$

Solution: Let $X_i = \sum_{j=1}^n(\delta_{ij}-p)^2$. We have $X_i = \sum_{j=1}^n\delta_{ij}^2 + np^2 - 2p\sum_{j=1}^n\delta_{ij}$. Define $N_i = \sum_{j=1}^n\delta_{ij} = \sum_{j=1}^n\delta_{ij}^2$, then $X_i = (1-2p)N_i + np^2$.

Suppose $p \geq \frac{1}{2}$, then for any i, we have $X_i \leq np^2 \leq pn$. Thus the result follows trivially. Now we assume $p < \frac{1}{2}$. It suffices to show that $\mathbb{E}\max_{i\leq n}N_i \leq Cpn$. By Bernstein's inequality, we have

$$\mathbb{P}\left(N_i \geq (1+t)np\right) \leq \exp\left(-\frac{t^2n^2p^2}{2np(1-p)+2tnp/3}\right).$$

Applying the union bound, we have

$$\mathbb{P}\left(\max_{i\le n} N_i \ge (1+t)np\right) \le n\exp\left(-\frac{t^2n^2p^2}{2np(1-p)+2tnp/3}\right).$$

When $t > 10$, we have $\frac{2}{3}t \ge 2(1-p)$ and $\log n \le pn \le tpn/10$, which gives

$$\mathbb{P}\left(\max_{i\le n} N_i \ge (1+t)np\right) \le n\exp\left(-\frac{tnp}{2}\right) \le \exp\left(-\frac{2tnp}{5}\right).$$

Therefore, for any $s > 20$,

$$\mathbb{P}\left(\frac{\max_{i\le n} N_i}{np} \ge s\right) \le \mathbb{P}\left(\frac{\max_{i\le n} N_i}{np} \ge (1+s/2)\right) \le \exp\left(-\frac{snp}{5}\right),$$

We can get the desired result by integrating the tail probability:

$$\mathbb{E}\frac{\max_{i\le n} N_i}{np} \le 20 + \int_{20}^{\infty} \exp\left(-\frac{snp}{5}\right) ds = 20 + \frac{5}{np}\exp(-4np) \le 25.$$

We finish the proof.

6.32 (Matrix completion with noisy observations) Extend matrix completion (Theorem 6.5.1) to noisy observations, where we are shown noisy versions $X_{ij} + \nu_{ij}$ of some entries of X. Here ν_{ij} are independent and mean zero subgaussian random variables representing noise.

Solution: We have the following result for matrix completion with noisy observations.

Proposition Let X be an $n \times n$ matrix of rank at most r. Let $Y_{ij} := \delta_{ij}(X_{ij} + \nu_{ij})$, where $\delta_{ij} \sim \text{Ber}(p)$ are independent and ν_{ij} are independent mean zero subgaussian random variables with $\max_{i,j} \|\nu_{ij}\|_{\psi_2} \le K$. Let $\hat{X}$ be a best rank r approximation to $p^{-1}Y$, where $p = m/n^2$ and m is the expected number of observed entries. If $m \ge n\log n$, then

$$\mathbb{E}\,\frac{1}{n}\left\|\hat{X} - X\right\|_F \le C\sqrt{\frac{rn\log n}{m}}\,\left(\|X\|_\infty + K\right).$$

Proof: The proof is similar to that of Theorem 6.5.1. Since both X and $\hat{X}$ have rank at most r, $\left\|\hat{X} - X\right\|_F \le \sqrt{2r}\,\left\|\hat{X} - X\right\|$. Moreover, because $\hat{X}$ is a best rank r approximation to $p^{-1}Y$ and X is admissible,

$$\left\|\hat{X} - X\right\| \le \left\|\hat{X} - p^{-1}Y\right\| + \left\|p^{-1}Y - X\right\|$$

$$\leq 2 \left\| p^{-1} Y - X \right\|$$
$$= \frac{2}{p} \left\| Y - pX \right\|.$$

Let $A := Y - pX$. Its entries $A_{ij} = (Y - pX)_{ij} = (\delta_{ij} - p)X_{ij} + \delta_{ij}\nu_{ij}$ are independent and mean zero. By Exercise 6.28,

$$\mathbb{E} \left\| A \right\| \leq C\sqrt{\log n}\, \mathbb{E} \max \left\{ \max_{i \leq n} \left\| A_{i:} \right\|_2, \max_{j \leq n} \left\| A_{:j} \right\|_2 \right\}.$$

It remains to bound the row and column norms. By Exercise 6.30,

$$\mathbb{E} \max_{i \leq n} \sum_{j=1}^{n} (\delta_{ij} - p)^2 \leq Cpn,$$

and the same bound holds for the columns. For the noise terms, $\delta_{ij}\nu_{ij}^2$ are sub-exponential random variables with mean at most CpK^2 and sub-exponential norm at most CK^2. Bernstein's inequality gives, for an absolute constant C_1,

$$\mathbb{E} \max_{i \leq n} \sum_{j=1}^{n} \delta_{ij}\nu_{ij}^2 \leq C_1 K^2 \left(pn + \log n \right) \leq 2C_1 K^2 pn = C_2 K^2 pn,$$

where $C_2 = 2C_1$, and the same bound holds for the columns. Therefore,

$$\mathbb{E} \max_{i \leq n} \left\| A_{i:} \right\|_2^2 \leq 2\mathbb{E} \max_{i \leq n} \sum_{j=1}^{n} (\delta_{ij} - p)^2 X_{ij}^2 + 2\mathbb{E} \max_{i \leq n} \sum_{j=1}^{n} \delta_{ij}\nu_{ij}^2$$
$$\leq Cpn \left(\left\| X \right\|_\infty^2 + K^2 \right),$$

and, by the same argument for columns,

$$\mathbb{E} \max_{j \leq n} \left\| A_{:j} \right\|_2^2 \leq Cpn \left(\left\| X \right\|_\infty^2 + K^2 \right).$$

Consequently,

$$\mathbb{E} \max \left\{ \max_{i \leq n} \left\| A_{i:} \right\|_2, \max_{j \leq n} \left\| A_{:j} \right\|_2 \right\} \leq C\sqrt{pn}\, \left(\left\| X \right\|_\infty + K \right).$$

Combining the estimates, we obtain

$$\mathbb{E} \left\| \hat{X} - X \right\|_F \leq C\sqrt{r}\, \frac{1}{p} \sqrt{pn \log n}\, \left(\left\| X \right\|_\infty + K \right).$$

Dividing by n and using $p = m/n^2$ completes the proof.

6.33 (Sums of independent unbounded random matrices) There are not many results that don't make any assumptions on the distribution of random matrices. Let's prove one – a version of matrix Bernstein inequality (5.17) for unbounded random matrices. Let $Z_1, \dots, Z_N$ be independent $n \times n$ positive semidefinite random matrices. Show that

$$S := \sum_{i=1}^N Z_i \quad \text{satisfies} \quad \mathbb{E}\|S - \mathbb{E}S\| \le C\left(\sqrt{\|\mathbb{E}S\| \cdot L} + L\right)$$

where

$$L = \log(n)\, \mathbb{E} \max_i \|Z_i\|.$$

Solution: Let $M = \max_i \|Z_i\|$. By symmetrization,

$$\mathbb{E}\,\|S - \mathbb{E}S\| \le 2\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i Z_i\right\|,$$

where $\varepsilon_1, \dots, \varepsilon_N$ are independent Rademacher random variables independent of $Z_1, \dots, Z_N$. Conditioning on $Z_1, \dots, Z_N$ and applying matrix Khintchine's inequality (Theorem 5.4.14) gives

$$\mathbb{E}\left\|\sum_{i=1}^N \varepsilon_i Z_i\right\| \le C_0\sqrt{\log n}\, \mathbb{E}\left\|\sum_{i=1}^N Z_i^2\right\|^{1/2}.$$

Since each Z_i is positive semidefinite, $Z_i^2 \preceq \|Z_i\|\, Z_i \preceq M Z_i$. Therefore

$$\sum_{i=1}^N Z_i^2 \preceq M \sum_{i=1}^N Z_i = MS,$$

and hence $\left\|\sum_{i=1}^N Z_i^2\right\|^{1/2} \le M^{1/2}\, \|S\|^{1/2}$. By Cauchy-Schwarz,

$$\mathbb{E}\left\|\sum_{i=1}^N Z_i^2\right\|^{1/2} \le \mathbb{E}\left(M^{1/2}\, \|S\|^{1/2}\right) \le (\mathbb{E}M)^{1/2}\, (\mathbb{E}\,\|S\|)^{1/2}.$$

Combining the last estimates and using $L = \log(n)\mathbb{E}M$, we get

$$\mathbb{E}\,\|S - \mathbb{E}S\| \le C_1\sqrt{L\, \mathbb{E}\,\|S\|}.$$

Let $x = \mathbb{E}\,\|S - \mathbb{E}S\|$ and $a = \|\mathbb{E}S\|$. Since $\mathbb{E}\,\|S\| \le \mathbb{E}\,\|S - \mathbb{E}S\| + \|\mathbb{E}S\| = x + a$, we have

$$x \le C_1\sqrt{L(x+a)}.$$

Using $\sqrt{u+v} \le \sqrt{u} + \sqrt{v}$, we get

$$x \le C_1\sqrt{Lx} + C_1\sqrt{La}.$$

The inequality $C_1\sqrt{Lx} \le x/2 + C_1^2 L/2$ gives

$$\frac{x}{2} \le \frac{C_1^2}{2}L + C_1\sqrt{La}.$$

Thus

$$x \le C_1^2 L + 2C_1\sqrt{La} \le C\left(\sqrt{\|\mathbb{E}S\| \cdot L} + L\right),$$

which completes the proof.

6.34 (Covariance estimation for unbounded distributions) Let's relax the boundedness assumption (5.20) in the general covariance estimation result (5.27). Let X be a random vector in $\mathbb{R}^n$ with $\Sigma = \mathbb{E}XX^{\mathsf{T}}$. Assume that for some $K \ge 1$,

$$\mathbb{E}\max_{i\le m}\|X_i\|_2^2 \le K^2\mathbb{E}\|X\|_2^2.$$

Let $X_1, \dots, X_m$ be i.i.d. copies of X, and $\Sigma_m = \frac{1}{m}\sum_{i=1}^m X_i X_i^{\mathsf{T}}$. Prove that

$$\mathbb{E}\|\Sigma_m - \Sigma\| \le C\left(\sqrt{\frac{K^2 r\log n}{m}} + \frac{K^2 r\log n}{m}\right)\|\Sigma\|,$$

where $r = \mathrm{tr}(\Sigma)/\|\Sigma\|$ is the effective rank of Σ.

Solution: We can get the desired result by applying the result of Exercise 6.33. Specifically, let $Z_i = X_i X_i^{\top}$ and $S = \sum_{i=1}^m Z_i$. Then $\Sigma_m = S/m$ and $\Sigma = \mathbb{E}Z_1$. Moreover, we have

$$\begin{aligned} L &= \log(n)\,\mathbb{E}\max_i\|Z_i\| \\ &= \log(n)\,\mathbb{E}\max_i\|X_i\|_2^2 \\ &\le K^2\log(n)\,\mathbb{E}\|X\|_2^2 \end{aligned}$$

$$= K^2 \log(n)\ \mathrm{tr}(\Sigma).$$

Applying the result of Exercise 6.33 gives

$$\mathbb{E}\|S - \mathbb{E}S\| \le C\left(\sqrt{\|\mathbb{E}S\| \cdot L} + L\right) = C\left(\sqrt{m\|\Sigma\| \cdot K^2 \log(n)\ \mathrm{tr}(\Sigma)} + K^2 \log(n)\ \mathrm{tr}(\Sigma)\right).$$

Dividing both sides by m gives

$$\mathbb{E}\|\Sigma_m - \Sigma\| \le C\left(\sqrt{\frac{K^2 \log(n)\ \mathrm{tr}(\Sigma)}{m}\|\Sigma\|} + \frac{K^2 \log(n)\ \mathrm{tr}(\Sigma)}{m}\right) = C\left(\sqrt{\frac{K^2 r \log n}{m}} + \frac{K^2 r \log n}{m}\right)\|\Sigma\|,$$

where $r = \mathrm{tr}(\Sigma)/\|\Sigma\|$ is the effective rank of Σ. This completes the proof.

6.35 Let $z_1, \dots, z_N$ be any vectors in a normed space, let $a = (a_1, \dots, a_N) \in \mathbb{R}^N$, and let $\varepsilon_1, \dots, \varepsilon_N$ be independent Rademacher random variables. Show that the following function is convex:

$$f(a) := \mathbb{E}\left\|\sum_{i=1}^N a_i \varepsilon_i z_i\right\|.$$

Solution: First, note that $\sum_{i=1}^N a_i \varepsilon_i z_i$ is a linear combination of the vectors z_i with coefficients $a_i \varepsilon_i$. Since the norm is convex, the function $\left\|\sum_{i=1}^N a_i \varepsilon_i z_i\right\|$ is convex in a. Therefore, $f(a)$ is convex as it is a weighted average of convex functions.

6.36 (Contraction principle for general distributions) Let's prove the following generalization of Theorem 6.6.1. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, and let $X_1, \dots, X_N$ be independent, mean-zero random vectors taking values in a normed space. Show that

$$\mathbb{E}\left\|\sum_{i=1}^N a_i X_i\right\| \le 4\|a\|_\infty \cdot \mathbb{E}\left\|\sum_{i=1}^N X_i\right\|.$$

Solution: By symmetrization, we have $\mathbb{E}\left\|\sum_{i=1}^N a_i X_i\right\| \le 2\mathbb{E}\left\|\sum_{i=1}^N a_i \varepsilon_i X_i\right\|$. Conditioning on $X_1, \dots, X_N$ and applying the contraction principle gives $\mathbb{E}_\varepsilon\left\|\sum_{i=1}^N a_i \varepsilon_i X_i\right\| \le 2\|a\|_\infty \mathbb{E}_\varepsilon\left\|\sum_{i=1}^N \varepsilon_i X_i\right\|$. Taking expectations over $X_1, \dots, X_N$ gives the desired result.

6.37 (Logarithmic factor is unavoidable in Gaussian symmetrization) Show that, in general, the factor $\sqrt{\log N}$ in Lemma 6.6.2 is optimal.

Solution: Let $X_i = \varepsilon_i e_i$ for $i = 1, \dots, N$, where ε_i are independent Rademacher random variables and e_i are the standard basis vectors in $\mathbb{R}^N$. Then we have $\mathbb{E}\left\|\sum_{i=1}^N a_i X_i\right\|_\infty = 1$. On the other hand, by Exercise 2.38, we have when N is sufficiently large,

$$\mathbb{E}\left\|\sum_{i=1}^N g_i \varepsilon_i X_i\right\|_\infty = \mathbb{E}\max_{i\le N}|g_i| \ge c\sqrt{\log N}.$$

We finish the proof.

7 Random Processes

7.1 (Covariance vs. increments) Consider a random process $(X_t)_{t\in T}$.

(a) Assuming that the zero random variable 0 belongs to the process, express the covariance function $\Sigma(t,s) = \mathbb{E}X_t X_s$ in terms of the metric $d(t,s) = \|X_t - X_s\|_{L_2}$.

(b) Do the same assuming that the process contains the random variable $-X_t$ whenever it contains X_t.

(c) Show by example that without any assumption, it may be impossible to recover the covariance function from the metric.

Solution: (a) By the definition of the metric, we have

$$\begin{aligned}\Sigma(t,s) &= \mathbb{E}X_t X_s \\ &= \frac{1}{2}\left(\mathbb{E}X_t^2 + \mathbb{E}X_s^2 - \mathbb{E}(X_t - X_s)^2\right) \\ &= \frac{1}{2}\left(d(t,0)^2 + d(s,0)^2 - d(t,s)^2\right).\end{aligned}$$

(b) Again, by the definition of the metric, we have

$$\begin{aligned}\Sigma(t,s) &= \mathbb{E}X_t X_s \\ &= \frac{1}{4}\left(\mathbb{E}(X_t + X_s)^2 - \mathbb{E}(X_t - X_s)^2\right) \\ &= \frac{1}{4}\left(d(t,s')^2 - d(t,s)^2\right).\end{aligned}$$

Here we use the notation s' to denote the index of the random variable $-X_s$.

(c) Let two independent standard normal random variables g and h. Consider the following two random processes:

$$X_t = tg,\ t \in (1,2) \quad \text{and} \quad Y_t = tg + h,\ t \in (1,2).$$

Then we have $d_X(t,s) = d_Y(t,s) = |t-s|$ for all $t, s \in (1,2)$. However, the covariance functions $\Sigma_X(t,s)$ and $\Sigma_Y(t,s)$ are different.

7.2 (Symmetrization for random processes) Let $X_1(t), \dots, X_N(t)$ be N independent, mean zero random processes indexed by points $t \in T$. Let $\varepsilon_1, \dots, \varepsilon_N$ be independent Rademacher random variables. Prove that

$$\frac{1}{2}\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^N \varepsilon_i X_i(t)\right| \le \mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^N X_i(t)\right| \le 2\mathbb{E}\sup_{t\in T}\left|\sum_{i=1}^N \varepsilon_i X_i(t)\right|.$$

Solution: Following the definition of $\max_{t\in T}$ in the HDP book, we can assume $|T| < \infty$ without loss of generality. Consider the normed space $V = (\mathbb{R}^T, \|\cdot\|_\infty)$. We can directly get the result by viewing the random processes as random elements in V and applying Lemma 6.3.2.

7.3 (Talagrand contraction principle) In this exercise, we get a first look at random processes, which we will dive into fully in the next chapter. Let $T \subset \mathbb{R}^n$ be a bounded set. Let $\phi_i : \mathbb{R} \to \mathbb{R}$ be contractions, i.e. Lipschitz functions with $\|\phi_i\|_{\text{Lip}} \le 1$. Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher random variables. Show that

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^n \varepsilon_i \phi_i(t_i) \le \mathbb{E}\sup_{t\in T}\sum_{i=1}^n \varepsilon_i t_i$$

by going through the following steps:

(a) First let $n = 2$. Consider a subset $T \subset \mathbb{R}^2$ and contraction $\phi : \mathbb{R} \to \mathbb{R}$, and check that

$$\sup_{t\in T}\big(t_1 + \phi(t_2)\big) + \sup_{t\in T}\big(t_1 - \phi(t_2)\big) \le \sup_{t\in T}(t_1 + t_2) + \sup_{t\in T}(t_1 - t_2).$$

(b) Use induction on n to complete proof.

Solution:

(a) Since T is a bounded subset of $\mathbb{R}^2$, for any continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, we have $\sup_{t\in T} f(t) = \sup_{t\in \bar{T}} f(t)$. Therefore, we can assume T is tight with out loss of generality. Since T is compact and the function $t \mapsto t_1 + \phi(t_2)$, $t \mapsto t_1 - \phi(t_2)$ are continuous, there exist $t^{(1)}, t^{(2)} \in T$ such that

$$\sup_{t\in T}\big(t_1 + \phi(t_2)\big) = t_1^{(1)} + \phi(t_2^{(1)}), \quad \sup_{t\in T}\big(t_1 - \phi(t_2)\big) = t_1^{(2)} - \phi(t_2^{(2)}).$$

Then we have

$$\begin{aligned}\sup_{t\in T}\big(t_1 + \phi(t_2)\big) + \sup_{t\in T}\big(t_1 - \phi(t_2)\big) &= t_1^{(1)} + \phi(t_2^{(1)}) + t_1^{(2)} - \phi(t_2^{(2)}) \\ &= (t_1^{(1)} + t_1^{(2)}) + (\phi(t_2^{(1)}) - \phi(t_2^{(2)})) \\ &\le (t_1^{(1)} + t_1^{(2)}) + \left|t_2^{(1)} - t_2^{(2)}\right|.\end{aligned}$$

The last inequality follows from the fact that ϕ is a contraction. Suppose $t_2^{(1)} \ge t_2^{(2)}$. Then

$$\begin{aligned}t_1^{(1)} + t_1^{(2)} + \left|t_2^{(1)} - t_2^{(2)}\right| &= t_1^{(1)} + t_1^{(2)} + (t_2^{(1)} - t_2^{(2)}) \\ &= (t_1^{(1)} + t_2^{(1)}) + (t_1^{(2)} - t_2^{(2)}) \\ &\le \sup_{t\in T}(t_1 + t_2) + \sup_{t\in T}(t_1 - t_2).\end{aligned}$$

By exactly the same argument, we can show that the inequality also holds when $t_2^{(1)} < t_2^{(2)}$.

(b) Directly applying the standard "induction on n" argument can be tricky, so we will adopt a different approach that shares the same idea. For $m = 0, \ldots, n$, define

$$A_m = \mathbb{E}\sup_{t\in T}\left(\sum_{i=1}^{m}\varepsilon_i t_i + \sum_{i=m+1}^{n}\varepsilon_i\phi_i(t_i)\right).$$

We show that $A_{m-1} \le A_m$ for every $m = 1, \ldots, n$. Condition on all Rademacher variables except ε_m, and write

$$a(t) = \sum_{i<m}\varepsilon_i t_i + \sum_{i>m}\varepsilon_i\phi_i(t_i).$$

Then

$$\begin{aligned}\mathbb{E}_{\varepsilon_m}\sup_{t\in T}\big(a(t) + \varepsilon_m\phi_m(t_m)\big) &= \frac{1}{2}\left[\sup_{t\in T}\big(a(t) + \phi_m(t_m)\big) + \sup_{t\in T}\big(a(t) - \phi_m(t_m)\big)\right] \\ &\le \frac{1}{2}\left[\sup_{t\in T}\big(a(t) + t_m\big) + \sup_{t\in T}\big(a(t) - t_m\big)\right]\end{aligned}$$

$$= \mathbb{E}_{\varepsilon_m} \sup_{t \in T} \big(a(t) + \varepsilon_m t_m\big),$$

where the inequality follows from part (a), applied to the set $\{(a(t), t_m) : t \in T\} \subset \mathbb{R}^2$. Taking expectation over the remaining Rademacher variables gives $A_{m-1} \le A_m$. Therefore, we can get $A_0 \le A_1 \le \cdots \le A_n$, which proves the desired result.

7.5 (Expressing a random walk as a canonical process) Express an N-step random walk of Example 7.1.3 with $Z_i \sim \mathcal{N}(0,1)$ as a canonical Gaussian process with some $T \subset \mathbb{R}^N$.

Solution: Let $T = \Big\{t_k = \sum_{i=1}^k e_i, k = 1, \ldots, N\Big\}$. Then we can express the N-step random walk as a canonical Gaussian process as $X_n = \langle Z, t_n \rangle$, where $Z = (Z_1, \ldots, Z_N)$ is a standard Gaussian vector in $\mathbb{R}^N$.

7.6 (Multivariate Gaussian integration by parts) Prove Lemma 7.2.4.

Solution: First, it suffices to prove the result for the case $\Sigma = I_n$. Suppose that for a standard Gaussian vector $g \sim \mathcal{N}(0, I_n)$, we have $\mathbb{E}gh(g) = \mathbb{E}\nabla h(g)$ for every differentiable function h. For any Gaussian vector $X \sim \mathcal{N}(0, \Sigma)$, write $X = \Sigma^{1/2} g$ and define $h(g) = f(\Sigma^{1/2} g)$. By the chain rule,

$$\mathbb{E}Xf(X) = \Sigma^{1/2}\mathbb{E}gh(g) = \Sigma^{1/2}\mathbb{E}\nabla h(g) = \Sigma^{1/2}\mathbb{E}(\Sigma^{1/2})^\top \nabla f(X) = \Sigma \mathbb{E}\nabla f(X),$$

where we used the symmetric square root of Σ.

Now we prove the result for standard Gaussian vector $g \sim \mathcal{N}(0, I_n)$. First, we can assume f is supported on a bounded set. Then we can write

$$\begin{aligned}
(\mathbb{E}\nabla f(g))_i &= \int \frac{\partial f}{\partial x_i}(x) \frac{1}{(2\pi)^{n/2}} e^{-\|x\|_2^2/2} dx \\
&= \int\int \frac{\partial f}{\partial x_i}(x) \frac{1}{(2\pi)^{1/2}} e^{-x_i^2/2} dx_i \frac{1}{(2\pi)^{(n-1)/2}} e^{-\|x_{-i}\|_2^2/2} dx_{-i} \\
&= -\int\int f(x) \frac{\partial}{\partial x_i}\left(\frac{1}{(2\pi)^{1/2}} e^{-x_i^2/2}\right) dx_i \frac{1}{(2\pi)^{(n-1)/2}} e^{-\|x_{-i}\|_2^2/2} dx_{-i} \\
&= -\int f(x)(-x_i) \frac{1}{(2\pi)^{n/2}} e^{-\|x\|_2^2/2} dx \\
&= \mathbb{E} g_i f(g).
\end{aligned}$$

For a general f, let χ_R be a smooth function such that $0 \le \chi_R \le 1$, $\chi_R = 1$ on $B(0,R)$, $\chi_R = 0$

outside $B(0,2R)$, and $\|\nabla\chi_R\|_2 \le C/R$. Apply the compactly supported case to $f_R = f\chi_R$ to obtain

$$\mathbb{E}g_i f_R(g) = \mathbb{E}\frac{\partial f_R}{\partial x_i}(g) = \mathbb{E}\left[\chi_R(g)\frac{\partial f}{\partial x_i}(g)\right] + \mathbb{E}\left[f(g)\frac{\partial \chi_R}{\partial x_i}(g)\right].$$

By dominated convergence, the left-hand side converges to $\mathbb{E}g_i f(g)$ and the first term on the right-hand side converges to $\mathbb{E}\frac{\partial f}{\partial x_i}(g)$. Moreover,

$$\mathbb{E}\left|f(g)\frac{\partial \chi_R}{\partial x_i}(g)\right| \le \frac{C}{R}\mathbb{E}|f(g)| \to 0.$$

Letting $R \to \infty$ gives $\mathbb{E}g_i f(g) = \mathbb{E}\frac{\partial f}{\partial x_i}(g)$ for every i, which completes the proof.

7.7 (Differentiating the expected log-partition function) In the proof of Sudakov–Fernique Theorem 7.2.8, one step left for you was to check that for $Z(u) = \sqrt{u}X + \sqrt{1-u}\,Y$, the log-partition function

$$f(x) = \frac{1}{\beta}\log\sum_{i=1}^{n} e^{\beta x_i}$$

satisfies

$$\frac{d}{du}\mathbb{E}f(Z(u)) \le 0 \qquad \text{for all } u \in [0,1].$$

Prove this by following these steps:

(a) Differentiate f and check that

$$\frac{\partial f}{\partial x_i} = \frac{e^{\beta x_i}}{\sum_k e^{\beta x_k}} =: p_i(x) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i \partial x_j} = \beta\big(\delta_{ij}p_i(x) - p_i(x)p_j(x)\big),$$

where δ_{ij} is the Kronecker delta, which equals 1 if $i = j$ and 0 otherwise.

(b) Use the Gaussian interpolation formula 7.2.5. Simplify the expression and deduce that

$$\frac{d}{du}\mathbb{E}f(Z(u)) = \frac{\beta}{4}\sum_{i\neq j}\big[\mathbb{E}(X_i - X_j)^2 - \mathbb{E}(Y_i - Y_j)^2\big]\,\mathbb{E}\big[p_i(Z(u))p_j(Z(u))\big].$$

(c) Note that this expression is non-positive by assumption.

Solution: (a) Trivial.

(b) Directly applying the Gaussian interpolation formula 7.2.5 gives

$$\begin{aligned}
\frac{d}{du}\mathbb{E}f(Z(u)) &= \frac{1}{2}\sum_{i,j=1}^{n}\left[\mathbb{E}X_iX_j - \mathbb{E}Y_iY_j\right]\mathbb{E}\frac{\partial^2 f}{\partial x_i \partial x_j}(Z(u)) \\
&= \frac{\beta}{2}\sum_{i,j=1}^{n}\left[\mathbb{E}X_iX_j - \mathbb{E}Y_iY_j\right]\mathbb{E}\big(\delta_{ij}p_i(Z(u)) - p_i(Z(u))p_j(Z(u))\big) \\
&= \frac{\beta}{2}\sum_{i=1}^{n}\left[\mathbb{E}X_i^2 - \mathbb{E}Y_i^2\right]\mathbb{E}\big[p_i(Z(u)) - p_i(Z(u))^2\big] \\
&\quad - \frac{\beta}{2}\sum_{i\neq j}\left[\mathbb{E}X_iX_j - \mathbb{E}Y_iY_j\right]\mathbb{E}\big[p_i(Z(u))p_j(Z(u))\big] \\
&= \frac{\beta}{2}\sum_{i\neq j}\left[\mathbb{E}X_i^2 - \mathbb{E}Y_i^2 - \mathbb{E}X_iX_j + \mathbb{E}Y_iY_j\right]\mathbb{E}\big[p_i(Z(u))p_j(Z(u))\big] \\
&= \frac{\beta}{4}\sum_{i\neq j}\left[\mathbb{E}X_i^2 + \mathbb{E}X_j^2 - 2\mathbb{E}X_iX_j - \mathbb{E}Y_i^2 - \mathbb{E}Y_j^2 + 2\mathbb{E}Y_iY_j\right]\mathbb{E}\big[p_i(Z(u))p_j(Z(u))\big] \\
&= \frac{\beta}{4}\sum_{i\neq j}\left[\mathbb{E}(X_i - X_j)^2 - \mathbb{E}(Y_i - Y_j)^2\right]\mathbb{E}\big[p_i(Z(u))p_j(Z(u))\big].
\end{aligned}$$

The fourth equality follows from $p_i - p_i^2 = p_i\sum_{j\neq i}p_j$.

7.8 (Gaussian contraction inequality) The following is a Gaussian version of Talagrand contraction principle from Exercise 7.3. Consider a bounded subset $T \subset \mathbb{R}^n$, and let $g_1, \dots, g_n$ be independent $\mathcal{N}(0,1)$ random variables. Let $\phi_i : \mathbb{R} \to \mathbb{R}$ be contractions, i.e. Lipschitz functions with $\|\phi_i\|_{\text{Lip}} \leq 1$. Use Sudakov–Fernique inequality to prove that

$$\mathbb{E}\sup_{t\in T}\sum_{i=1}^{n} g_i\phi_i(t_i) \leq \mathbb{E}\sup_{t\in T}\sum_{i=1}^{n} g_i t_i.$$

Solution: Let $X_t = \sum_{i=1}^n g_i\phi_i(t_i)$ and $Y_t = \sum_{i=1}^n g_i t_i$. Since $\|\phi_i\|_{\text{Lip}} \leq 1$, we have

$$\mathbb{E}(X_t - X_s)^2 = \sum_{i=1}^{n}(\phi_i(t_i) - \phi_i(s_i))^2 \leq \sum_{i=1}^{n}(t_i - s_i)^2 = \mathbb{E}(Y_t - Y_s)^2.$$

By Sudakov–Fernique inequality, we have

$$\mathbb{E}\sup_{t\in T} X_t \leq \mathbb{E}\sup_{t\in T} Y_t.$$

7.9 (Gordon inequality) Prove Gordon inequality (Theorem 7.2.9) under the additional equal variances assumption:

$$\mathbb{E}X_{ut}^2 = \mathbb{E}Y_{ut}^2 \qquad \text{for all } u, t.$$

Like with the Sudakov–Fernique inequality, the equal variances assumption can be dropped from Gordon inequality too. We won't prove this here.

Solution: WLOG, we assume that $U \times T$ is finite. We may also assume that X and Y are independent. Consider the interpolation

$$Z(\lambda) = \sqrt{\lambda}X + \sqrt{1-\lambda}\,Y, \qquad \lambda \in [0,1].$$

Let $h_\tau : \mathbb{R} \to [0,1]$ be a twice-differentiable, non-increasing approximation of $\mathbf{1}_{(-\infty,\tau)}$, and, as suggested, define

$$f(x) = \prod_{u\in U}\left(1 - \prod_{t\in T} h_\tau(x_{ut})\right).$$

If $u \neq v$, then

$$\frac{\partial^2 f(x)}{\partial x_{ut}\partial x_{vs}} = h_\tau'(x_{ut})h_\tau'(x_{vs})\prod_{r\neq t} h_\tau(x_{ur})\prod_{r\neq s} h_\tau(x_{vr})\prod_{w\neq u,v}\left(1 - \prod_{r\in T} h_\tau(x_{wr})\right) \geq 0.$$

If $t \neq s$, then

$$\frac{\partial^2 f(x)}{\partial x_{ut}\partial x_{us}} = -h_\tau'(x_{ut})h_\tau'(x_{us})\prod_{r\neq t,s} h_\tau(x_{ur})\prod_{w\neq u}\left(1 - \prod_{r\in T} h_\tau(x_{wr})\right) \leq 0.$$

Moreover, by the equal variances assumption and the two increment conditions,

$$\mathbb{E}X_{ut}X_{vs} - \mathbb{E}Y_{ut}Y_{vs} = \frac{1}{2}\left[\mathbb{E}(Y_{ut} - Y_{vs})^2 - \mathbb{E}(X_{ut} - X_{vs})^2\right].$$

Thus, this covariance difference is zero on the diagonal, nonnegative when $u = v$ and $t \neq s$, and nonpositive when $u \neq v$. Thus, Lemma 7.2.5 gives

$$\frac{d}{d\lambda}\mathbb{E}f(Z(\lambda)) = \frac{1}{2}\sum_{u,v\in U}\sum_{t,s\in T}\left(\mathbb{E}X_{ut}X_{vs} - \mathbb{E}Y_{ut}Y_{vs}\right)\mathbb{E}\frac{\partial^2 f}{\partial x_{ut}\partial x_{vs}}(Z(\lambda)) \leq 0.$$

Consequently,

$$\mathbb{E}f(X) \leq \mathbb{E}f(Y).$$

Letting $h_\tau \to \mathbf{1}_{(-\infty,\tau)}$ and using bounded convergence, we obtain

$$\mathbb{P}\left\{\inf_{u\in U}\sup_{t\in T} X_{ut} \geq \tau\right\} \leq \mathbb{P}\left\{\inf_{u\in U}\sup_{t\in T} Y_{ut} \geq \tau\right\}.$$

The argument works for every $\tau \in \mathbb{R}$, so integrating the tails also gives

$$\mathbb{E}\inf_{u\in U}\sup_{t\in T} X_{ut} \leq \mathbb{E}\inf_{u\in U}\sup_{t\in T} Y_{ut}.$$

7.10 (Frobenius distance between rank-one matrices) This bound was used in the proof of Theorem 7.3.1. For unit vectors $u, w \in \mathbb{R}^{n-1}$ and unit vectors $v, z \in \mathbb{R}^{m-1}$, show that

$$\left\|uv^\top - wz^\top\right\|_F^2 \leq \|u-w\|_2^2 + \|v-z\|_2^2.$$

Solution: We add and subtract $wv^\top$ to get

$$\begin{aligned}\left\|uv^\top - wv^\top + wv^\top - wz^\top\right\|_F^2 &= \left\|(u-w)v^\top + w(v-z)^\top\right\|_F^2 \\ &= \left\|(u-w)v^\top\right\|_F^2 + \left\|w(v-z)^\top\right\|_F^2 + 2\operatorname{tr}(v(u-w)^\top w(v-z)^\top).\end{aligned}$$

Since u, v, w, z are unit vectors,

$$\begin{aligned}\operatorname{tr}(v(u-w)^\top w(v-z)^\top) &= \operatorname{tr}(vu^\top wv^\top - vu^\top wz^\top - vw^\top wv^\top + vw^\top wz^\top) \\ &= \operatorname{tr}(vu^\top wv^\top) - \operatorname{tr}(vu^\top wz^\top) - \operatorname{tr}(vw^\top wv^\top) + \operatorname{tr}(vw^\top wz^\top) \\ &= \operatorname{tr}(u^\top w) - \operatorname{tr}(vu^\top wz^\top) - 1 + \operatorname{tr}(vz^\top) \\ &= \langle u, w\rangle - \langle u, w\rangle\langle v, z\rangle - 1 + \langle v, z\rangle \\ &= (1 - \langle u, w\rangle)(\langle v, z\rangle - 1) \\ &\leq 0.\end{aligned}$$

Therefore, we have

$$\left\|uv^\top - wz^\top\right\|_F^2 \leq \left\|(u-w)v^\top\right\|_F^2 + \left\|w(v-z)^\top\right\|_F^2 = \langle u-w, v\rangle^2 + \langle w, v-z\rangle^2 \leq \|u-w\|_2^2 + \|v-z\|_2^2.$$

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7.11 (Norms of GOE random matrices) Adapt the arguments in Section 7.3 to the symmetric case. Consider an $n \times n$ Gaussian random matrix A with independent normal entries on and above the diagonal, where the diagonal entries are $\mathcal{N}(0, 2)$ and the off-diagonal entries are $\mathcal{N}(0, 1)$. (This is the Gaussian orthogonal ensemble from Exercise 3.19.)

(a) Show the expectation bound:

$$\mathbb{E}\,\|A\| \le 2\sqrt{n}.$$

(b) Boost this to a high-probability bound: for any $t \ge 0$,

$$\mathbb{P}\left\{\|A\| \ge 2\sqrt{n} + t\right\} \le 2\exp(-ct^2).$$

Solution: (a) Let G be a Ginibre matrix, then we can write $A = (G + G^\top)/\sqrt{2}$ by Exercise 3.19. Therefore,

$$\begin{aligned}
\|A\| &= \sup_{u \in S^{n-1}} \left| u^\top A u \right| \\
&= \sup_{u \in S^{n-1}} \left| u^\top (G + G^\top) u / \sqrt{2} \right| \\
&= \sqrt{2} \sup_{u \in S^{n-1}} \left| (Gu)^\top u \right|.
\end{aligned}$$

Following the same proof strategy as in Theorem 7.3.1, we consider a Gaussian process $X_u = (Gu)^\top u$ indexed by $u \in T \coloneqq S^{n-1}$, and $\mathbb{E}\,\|A\| = \sqrt{2}\mathbb{E}\sup_{u \in T} X_u$. And

$$\mathbb{E}(X_u - X_v)^2 = \mathbb{E}\left(\sum_{i,j} G_{ij}^2 (u_i u_j - v_i v_j)^2\right)^2 \le 2\,\|u - v\|_2^2\,.$$

On the other hand, let $Y_u = \sqrt{2} g^\top u$ be another Gaussian process indexed by $u \in T$, where $g \sim \mathcal{N}(0, I_n)$. Then we have

$$\mathbb{E}(Y_u - Y_v)^2 = 2\,\|u - v\|_2^2\,.$$

Note that $\mathbb{E}(X_u - X_v)^2 \le \mathbb{E}(Y_u - Y_v)^2$ for all $u, v \in T$, by Sudakov-Fernique inequality, we have

$$\mathbb{E}\sup_{u \in T} X_u \le \mathbb{E}\sup_{u \in T} Y_u = \sqrt{2}\mathbb{E}\,\|g\| \le \sqrt{2n}.$$

Therefore, we have $\mathbb{E}\,\|A\| \le 2\sqrt{n}$.

(b) Write $A_{ii} = \sqrt{2}g_{ii}$ and $A_{ij} = A_{ji} = g_{ij}$ for $i < j$, where $(g_{ij})_{i \le j}$ is a standard Gaussian vector. For two such vectors g, g', the corresponding matrices A, A' satisfy

$$\left| \|A\| - \|A'\| \right| \le \|A - A'\| \le \|A - A'\|_F = \sqrt{2} \|g - g'\|_2 .$$

Thus $g \mapsto \|A\|$ is $\sqrt{2}$-Lipschitz. By Gaussian concentration (Theorem 5.2.3) and part (a),

$$\mathbb{P}\left\{ \|A\| \ge 2\sqrt{n} + t \right\} \le \mathbb{P}\left\{ \|A\| \ge \mathbb{E}\|A\| + t \right\} \le 2\exp(-ct^2).$$

7.12 (The norm of a Gaussian vector) For $g \sim \mathcal{N}(0, I_n)$, show that the function

$$f(n) = \mathbb{E}\|g\|_2 - \sqrt{n}$$

is increasing on $[C, \infty)$, where C is a suitable absolute constant.

Solution: Since $\|g\|_2^2 \sim \chi_n^2$, its central moments satisfy

$$\mathbb{E}(\|g\|_2^2 - n)^2 = 2n, \qquad \mathbb{E}(\|g\|_2^2 - n)^3 = 8n, \qquad \mathbb{E}(\|g\|_2^2 - n)^4 = 12n^2 + 48n.$$

Substituting $u = \|g\|_2^2/n - 1$ into

$$\sqrt{1+u} = 1 + \frac{u}{2} - \frac{u^2}{8} + \frac{u^3}{16} - \frac{5u^4}{128} + O(|u|^5)$$

and taking expectations gives

$$\begin{aligned} \frac{\mathbb{E}\|g\|_2}{\sqrt{n}} &= 1 - \frac{1}{8}\cdot\frac{2}{n} + \frac{1}{16}\cdot\frac{8}{n^2} - \frac{5}{128}\left(\frac{12}{n^2} + \frac{48}{n^3}\right) + O(n^{-5/2}) \\ &= 1 - \frac{1}{4n} + \frac{1}{32n^2} + O(n^{-5/2}). \end{aligned}$$

Consequently,

$$f(n) = \mathbb{E}\|g\|_2 - \sqrt{n} = -\frac{1}{4\sqrt{n}} + \frac{1}{32n^{3/2}} + O(n^{-2}).$$

Therefore,

$$f(n+1) - f(n) = \frac{1}{8n^{3/2}} + O(n^{-2}) > 0$$

for all sufficiently large n, which proves the claim.

7.13 (Smallest singular values of Gaussian matrices) Let's use the Gordon inequality (Theorem 7.2.9) to get sharp bounds on the smallest singular value of an $m \times n$ random matrix A with independent $\mathcal{N}(0,1)$ entries. For $m \geq n \geq C$ (where C is a suitable absolute constant), show that:

(a) (Expectation)

$$\mathbb{E} s_n(A) \geq \sqrt{m} - \sqrt{n};$$

(b) (High probability)

$$\mathbb{P}\left\{ s_n(A) \leq \sqrt{m} - \sqrt{n} - t \right\} \leq 2\exp(-ct^2) \qquad \text{for any } t \geq 0.$$

Solution: (a) Let $g \sim \mathcal{N}(0, I_n)$ and $h \sim \mathcal{N}(0, I_m)$ be independent, and define

$$X_{u,v} = \langle g, u \rangle + \langle h, v \rangle, \qquad Y_{u,v} = \langle Au, v \rangle, \qquad (u,v) \in S^{n-1} \times S^{m-1}.$$

For fixed u and $v, w \in S^{m-1}$,

$$\mathbb{E}(X_{u,v} - X_{u,w})^2 = \|v - w\|_2^2 = \mathbb{E}(Y_{u,v} - Y_{u,w})^2.$$

Moreover, for $u, z \in S^{n-1}$ and $v, w \in S^{m-1}$,

$$\mathbb{E}(X_{u,v} - X_{z,w})^2 - \mathbb{E}(Y_{u,v} - Y_{z,w})^2 = 2(1 - \langle u, z \rangle)(1 - \langle v, w \rangle) \geq 0.$$

Thus, Gordon inequality gives

$$\begin{aligned}
\mathbb{E} s_n(A) &= \mathbb{E} \min_{u \in S^{n-1}} \max_{v \in S^{m-1}} Y_{u,v} \\
&\geq \mathbb{E} \min_{u \in S^{n-1}} \max_{v \in S^{m-1}} X_{u,v} \\
&= \mathbb{E} \|h\|_2 - \mathbb{E} \|g\|_2 \\
&= \sqrt{m} - \sqrt{n} + f(m) - f(n) \\
&\geq \sqrt{m} - \sqrt{n},
\end{aligned}$$

where the last inequality follows from Exercise 7.12 and $m \geq n \geq C$.

(b) By Weyl inequality, for any $m \times n$ matrices A and B,

$$|s_n(A) - s_n(B)| \le \|A - B\| \le \|A - B\|_F .$$

Thus, $A \mapsto s_n(A)$ is 1-Lipschitz as a function of the mn independent standard Gaussian entries of A. Gaussian concentration and part (a) yield

$$\begin{aligned}\mathbb{P}\left\{s_n(A) \le \sqrt{m} - \sqrt{n} - t\right\} &\le \mathbb{P}\left\{s_n(A) \le \mathbb{E}s_n(A) - t\right\} \\ &\le 2\exp(-ct^2),\end{aligned}$$

which proves the claim.

7.14 (Sudakov inequality for non-compact sets) Show that if (T, d) is not relatively compact, i.e. $\mathcal{N}(T, d, \varepsilon) = \infty$ for some $\varepsilon > 0$, then $\mathbb{E}\sup_{t\in T} X_t = \infty$.

Solution: If (T, d) is not relatively compact, then there exists $\varepsilon_0 > 0$ such that $\mathcal{N}(T, d, \varepsilon_0) = \infty$. This means that for any $N \in \mathbb{N}$, we can find a subset $T_N \subset T$ of size N such that the points in T_N are ε_0-separated with respect to the metric d. Then we restrict the process X_t to the finite set T_N. By Sudakov inequality, we have

$$\mathbb{E}\sup_{t\in T_N} X_t \ge c\varepsilon_0\sqrt{\log N},$$

where $c > 0$ is an absolute constant. Since N can be arbitrarily large, and it always holds that $\mathbb{E}\sup_{t\in T} X_t \ge \mathbb{E}\sup_{t\in T_N} X_t$, we conclude that $\mathbb{E}\sup_{t\in T} X_t = \infty$.

7.17 (Gaussian width of the ℓ_p ball) Let $1 \le p \le \infty$. Consider the unit ball of the ℓ_p norm in $\mathbb{R}^n$, i.e. $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$. Prove that

$$w(B_p^n) \asymp \begin{cases} \sqrt{p'}\, n^{1/p'}, & p' \le \log n, \\ \sqrt{\log n}, & p' \ge \log n, \end{cases}$$

where p' is the conjugate exponent for p, as in Hölder inequality (1.22).

Solution: The result follows as a direct corollary of Exercise 3.5 and Exercise 3.6 by noting that $w(B_p^n) = \mathbb{E}\|g\|_{p'}$ for $g \sim \mathcal{N}(0, I_n)$.

7.18 (Nuclear norm) For an $n \times n$ matrix A, the operator norm $\|A\|$ is the ℓ_∞ norm of the singular values, and the Frobenius norm $\|A\|_F$ is the ℓ_2 norm (see Lemma 4.1.11). Define the nuclear norm as the ℓ_1 norm of the singular values:

$$\|A\|_* := \sum_{i=1}^{n} s_i(A).$$

(a) Just like ℓ_1 and ℓ_∞ norms are dual for vectors (see Exercise 1.19(b)), prove that the nuclear and operator norms are dual for matrices:

$$\|A\|_* = \max\{\langle A, B\rangle : \|B\| \le 1\}.$$

(b) Conclude that the nuclear norm is indeed a norm.

Solution: (a) WLOG, we assume A is diagonal with singular values $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A) \ge 0$. Otherwise, we can write $A = U\Sigma V^\top$ for some orthogonal matrices U, V and a diagonal matrix Σ with the singular values of A on the diagonal. Then we have $\|A\|_\star = \|\Sigma\|_*$ and for any B with $\|B\| \le 1$,

$$\langle A, B\rangle = \langle U\Sigma V^\top, B\rangle = \langle \Sigma, V^\top BU\rangle,$$

where $V^\top BU$ is also a matrix with operator norm at most 1. Thus, it suffices to prove the claim for diagonal matrices. For any B with $\|B\| \le 1$,

$$\langle A, B\rangle = \sum_{i=1}^{n} s_i(A)B_{ii} \le \sum_{i=1}^{n} s_i(A)\,\|B\| \le \|A\|_*.$$

On the other hand, let B be the diagonal matrix with $B_{ii} = \operatorname{sign}(s_i(A))$. Then $\|B\| = 1$ and

$$\langle A, B\rangle = \sum_{i=1}^{n} s_i(A)B_{ii} = \sum_{i=1}^{n} s_i(A) = \|A\|_*.$$

(b) It is trivial to check that $\|A\|_\star \ge 0$ and $\|A\|_\star = 0$ if and only if $A = 0$. Now we check the triangle inequality. For any A_1, A_2,

$$\begin{aligned}
\|A_1 + A_2\|_\star &= \max\{\langle A_1 + A_2, B\rangle : \|B\| \le 1\} \\
&= \max\{\langle A_1, B\rangle + \langle A_2, B\rangle : \|B\| \le 1\} \\
&\le \max\{\langle A_1, B\rangle : \|B\| \le 1\} + \max\{\langle A_2, B\rangle : \|B\| \le 1\}
\end{aligned}$$

$$= \|A_1\|_\star + \|A_2\|_\star .$$

7.19 (Gaussian width of matrices with bounded operator norm) Show that the unit ball of the operator norm for $n \times n$ matrices,

$$T := \{B : \|B\| \le 1\}, \qquad \text{satisfies} \qquad w(T) \asymp n^{3/2}.$$

Solution: Let G be an $n \times n$ matrix with independent $\mathcal{N}(0,1)$ entries. By Exercise 7.18 we have

$$w(T) = \mathbb{E} \sup_{B \in T} \langle G, B \rangle = \mathbb{E} \|G\|_* = \mathbb{E} \sum_{i=1}^{n} s_i(G).$$

For the upper bound, since $s_i(G) \le s_1(G) = \|G\|$ for every i, Remark 4.4.4 gives $w(T) \le n\mathbb{E}\|G\| \le Cn^{3/2}$.

For the lower bound, Exercise 4.47 gives

$$\mathbb{P}\left\{ s_k(G) < \sqrt{n} - C(\sqrt{k} + t) \right\} \le 2\exp(-t^2), \qquad t \ge 0.$$

Integrating this tail bound, we obtain $\mathbb{E}s_k(G) \ge \sqrt{n} - C\sqrt{k} - C'$. Choose $k = \lfloor cn \rfloor$, where $c > 0$ is a sufficiently small absolute constant. Then $\mathbb{E}s_k(G) \ge c'\sqrt{n}$ for all sufficiently large n. Since $s_i(G) \ge s_k(G)$ for every $i \le k$, it follows that

$$w(T) = \mathbb{E} \sum_{i=1}^{n} s_i(G) \ge \sum_{i=1}^{k} \mathbb{E}s_i(G) \ge k\mathbb{E}s_k(G) \ge c''n^{3/2}.$$

7.20 (Gaussian width vs. Gaussian complexity) For any bounded set $T \subset \mathbb{R}^n$ and a point $y \in T$, show that

$$\gamma(T) \asymp w(T) + \|y\|_2 .$$

This is one of the statements in Proposition 7.5.11, which was left unproved.

Solution: For any fixed $y \in T$, we have

$$\begin{aligned}
\gamma(T) &= \mathbb{E} \sup_{x \in T} |\langle g, x \rangle| \\
&\le \mathbb{E} \sup_{x \in T} |\langle g, x - y \rangle| + \mathbb{E} |\langle g, y \rangle| \\
&\le \mathbb{E} \sup_{x, y' \in T} \left| \langle g, x - y' \rangle \right| + \mathbb{E} |\langle g, y \rangle|
\end{aligned}$$

$$= \gamma(T - T) + \mathbb{E}\,|\langle g, y\rangle|$$
$$= 2w(t) + \mathbb{E}\,|\langle g, y\rangle|\,.$$

The last equality follows from Lemma 7.5.11 (a). The upper bound follows by noting that $\mathbb{E}\,|\langle g, y\rangle| = \sqrt{2/\pi}\,\|y\|_2$. For the lower bound, we have $\gamma(T) \geq w(T)$ and $\gamma(T) \geq \mathbb{E}\,|\langle g, y\rangle| = \sqrt{2/\pi}\,\|y\|_2$. Therefore, $\gamma(T) \gtrsim \max\{w(T), \|y\|_2\} \asymp w(T) + \|y\|_2$.

7.21 (Effective dimension vs. algebraic dimension) We introduced the effective dimension of a bounded set $T \subset \mathbb{R}^n$ in Definition 7.5.12.

(a) Prove that the effective dimension is bounded by the linear-algebraic dimension: $d(T) \leq \dim(T)$.

(b) Show that the equality holds when T is a Euclidean ball in a subspace of $\mathbb{R}^n$.

Solution: (a) WLOG, we assume that $\operatorname{diam}(T) = 1$. Suppose $\dim(T) = k$, then let $\{v_1, \ldots, v_k\}$ be an orthonormal basis of $\operatorname{span}(T - T)$. Also, for any $x \in T - T$, we have $x = \sum_{i=1}^k \alpha_i v_i$ for some $\alpha_i \in \mathbb{R}$ and $\|\alpha\|_2 \leq 1$ as $\operatorname{diam}(T) \leq 1$. Suppose $g \sim \mathcal{N}(0, I_n)$ and $x = \sum_{i=1}^k \alpha_i v_i$, then we have

$$\begin{aligned}
\mathbb{E} \sup_{x \in T-T} \langle g, x\rangle^2 &\leq \mathbb{E} \sup_{\|\alpha\|_2 \leq 1} \left(\sum_{i=1}^k \alpha_i \langle g, v_i\rangle\right)^2 \\
&= \mathbb{E} \sup_{\|\alpha\|_2 \leq 1} \left\langle \alpha, (\langle g, v_1\rangle, \ldots, \langle g, v_k\rangle)^\top \right\rangle^2 \\
&= \mathbb{E} \left\| (\langle g, v_1\rangle, \ldots, \langle g, v_k\rangle)^\top \right\|_2^2 \\
&= k.
\end{aligned}$$

Thus, $d(T) \leq k = \dim(T)$.

(b) If T is a Euclidean ball in a k-dimensional subspace, then $T - T$ is the Euclidean ball in $\operatorname{span}(T - T)$ appearing in part (a). Thus, the inequality in part (a) becomes an equality, so $d(T) = k = \dim(T)$.

7.23 (Ellipsoids) An ellipsoid is just a linear transformation of the unit ball B_2^n via some $m \times n$ matrix A, i.e. the set $A(B_2^n)$.

(a) (Gaussian width) Show that

$$w\big(A(B_2^n)\big) \asymp \|A\|_F\,,$$

where the notation $\asymp$ hides positive absolute constant factors.

(b) (Effective dimension) Show that

$$d\big(A(B_2^n)\big) = r(A^\top A) = s(A),$$

where $d(\cdot)$ is the effective dimension, $r(\cdot)$ is the effective rank and $s(\cdot)$ is the stable rank (see Definition 7.5.12 and Remarks 5.6.3, 5.6.4).

Solution: (a) Let $g \sim \mathcal{N}(0, I_m)$, then

$$\begin{aligned} w\big(A(B_2^n)\big) &= \mathbb{E} \sup_{x \in B_2^n} \langle g, Ax \rangle \\ &= \mathbb{E} \sup_{x \in B_2^n} \langle A^\top g, x \rangle \\ &= \mathbb{E} \left\| A^\top g \right\|_2 . \end{aligned}$$

Write the SVD of A as $A = U\Sigma V^\top$, then

$$\mathbb{E} \left\| A^\top g \right\|_2 = \mathbb{E} \left\| \Sigma g \right\|_2 \asymp \left\| \Sigma \right\|_F = \left\| A \right\|_F .$$

Here we use the fact that orthogonal transformation preserve norms.

(b) For any $y \in A(B_2^n) - A(B_2^n)$, we have $y = Ax$ for some $x \in B_2^n - B_2^n = 2B_2^n$. Then $\|y\|_2 = \|Ax\|_2 = 2\|A\|$. Thus, $\operatorname{diam}(A(B_2^n)) = 2\|A\|$. On the other hand, suppose $g \sim \mathcal{N}(0, I_n)$,

$$\mathbb{E} \sup_{x \in A(B_2^n) - A(B_2^n)} \langle g, x \rangle^2 = \mathbb{E} \sup_{x \in 2B_2^n} \langle g, Ax \rangle^2 = 4\mathbb{E} \left\| A^\top g \right\|_2^2 = 4\|A\|_F^2 .$$

Therefore, we have

$$d\big(A(B_2^n)\big) = \frac{\|A\|_F^2}{\|A\|^2} = \frac{\operatorname{tr}(A^\top A)}{\|A^\top A\|} = r(A^\top A) = s(A).$$

7.24 (Modeling a random vector on the sphere) Let z be a fixed vector on the unit sphere S^{m-1}, and let B be an $n \times m$ matrix consisting of the first m columns of a random orthogonal matrix $U \sim \operatorname{Unif}(O(n))$. Check that

$$Bz \sim \operatorname{Unif}(S^{n-1}).$$

Solution: It is straightforward to check that Bz is a unit vector. Therefore, it suffices to show that Bz is invariant under orthogonal transformations. Let $Q \in O(n)$ be an arbitrary orthogonal matrix, we show that we always have $QBz \overset{d}{=} Bz$. As $U \sim \mathrm{Unif}(O(n))$, we have $QU \sim \mathrm{Unif}(O(n))$ as well. Thus, we can write

$$QBz = QU \begin{pmatrix} z \\ 0 \end{pmatrix} \overset{d}{=} U \begin{pmatrix} z \\ 0 \end{pmatrix} = Bz.$$

7.25 (Gaussian projections of sets) Prove a version of Theorem 7.6.1 for an $m \times n$ Gaussian random matrix G with independent $\mathcal{N}(0,1)$ entries. Specifically, show that for any bounded set $T \subset \mathbb{R}^n$, we have

$$\mathbb{E}\,\mathrm{diam}(GT) \asymp w(T) + \sqrt{m}\,\mathrm{diam}(T).$$

Here $w(T)$ is the Gaussian width of T.

Solution: WLOG, we assume that $\mathrm{diam}(T) = 1$. For the upper bound, we have

$$\mathbb{E}\,\mathrm{diam}(GT) = \mathbb{E} \sup_{x \in T-T} \|Gx\|_2 = \mathbb{E} \sup_{x \in T-T} \max_{z \in S^{m-1}} \langle Gx, z \rangle.$$

Choose a $1/2$-net $\mathcal{N}$ of S^{m-1} such that $|\mathcal{N}| \le 5^m$. Then we have

$$\mathbb{E}\,\mathrm{diam}(GT) \le 2\mathbb{E} \sup_{x \in T-T} \max_{z \in \mathcal{N}} \langle Gx, z \rangle$$

We first fix $z \in \mathcal{N}$ and establish a bound for $\sup_{x \in T-T} \langle Gx, z \rangle$. We have

$$\sup_{x \in T-T} \langle Gx, z \rangle = \sup_{x \in T-T} \langle G^\top z, x \rangle \overset{d}{=} \sup_{x \in T-T} \langle g, x \rangle,$$

where $g \sim \mathcal{N}(0, I_n)$ is a standard Gaussian vector in $\mathbb{R}^n$. Thus, we have

$$\mathbb{E} \sup_{x \in T-T} \langle Gx, z \rangle = \mathbb{E} \sup_{x \in T-T} \langle g, x \rangle = w(T-T) = 2w(T).$$

Since $\mathrm{diam}(T) = 1$, the function $g \mapsto \sup_{x \in T-T} \langle g, x \rangle$ is 1-Lipschitz. Therefore,

$$\mathbb{P}\left(\sup_{x \in T-T} \langle Gx, z \rangle \ge 2w(T) + t \right) = \mathbb{P}\left(\sup_{x \in T-T} \langle g, x \rangle \ge \mathbb{E} \sup_{x \in T-T} \langle g, x \rangle + t \right) \le 2\exp(-ct^2).$$

By the union bound, we have

$$\mathbb{P}\left(\sup_{x\in T-T}\max_{z\in\mathcal{N}}\langle Gx,z\rangle\geq 2w(T)+t\right)\leq 5^m\cdot 2\exp(-ct^2).$$

Choosing $t=C\sqrt{m}+s$ with C large enough, we have

$$\mathbb{P}\left(\sup_{x\in T-T}\max_{z\in\mathcal{N}}\langle Gx,z\rangle\geq 2w(T)+C\sqrt{m}+s\right)\leq 2\exp(-cs^2),$$

and further

$$\mathbb{P}\left(\sup_{x\in T-T}\max_{z\in S^{m-1}}\langle Gx,z\rangle\geq 4w(T)+C'\sqrt{m}+s\right)\leq 2\exp(-cs^2),$$

Therefore,

$$\begin{aligned}\mathbb{E}\operatorname{diam}(GT)&\leq 4w(T)+C'\sqrt{m}+\mathbb{E}\left(\operatorname{diam}(GT)-4w(T)-C'\sqrt{m}\right)_+\\&\leq 4w(T)+C'\sqrt{m}+\int_0^\infty 2\exp(-ct^2)dt\\&\lesssim w(T)+\sqrt{m}.\end{aligned}$$

We have established the upper bound. On the other hand, for the lower bound, the calculation above gives $\mathbb{E}\operatorname{diam}(GT)\geq 2w(T)$. Moreover, for every fixed $x\in T-T$, we have $Gx\overset{d}{=}\|x\|_2 h$, where $h\sim\mathcal{N}(0,I_m)$. Thus,

$$\mathbb{E}\operatorname{diam}(GT)\geq\sup_{x\in T-T}\mathbb{E}\|Gx\|_2=\mathbb{E}\|h\|_2\sup_{x\in T-T}\|x\|_2\gtrsim\sqrt{m}\operatorname{diam}(T).$$

Combining these two bounds and rescaling completes the proof.

7.26 (Random projections of sets: a lower bound) Show the lower bound in Theorem 7.6.1: for any bounded set $T\subset\mathbb{R}^n$,

$$\mathbb{E}\operatorname{diam}(PT)\geq c\left[w_s(T)+\sqrt{\frac{m}{n}}\operatorname{diam}(T)\right].$$

Solution: WLOG, we assume that $\operatorname{diam}(T)=1$. Following the same argument as in the proof of Theorem 7.6.1, we define Q as the first m columns of a random orthogonal matrix $U\sim\operatorname{Unif}(O(n))$. Then $P=QQ^\top$. Since $Q^\top Q=I_m$, we have $\|Px\|_2=\left\|QQ^\top x\right\|_2=\left\|Q^\top x\right\|_2$ for every $x\in\mathbb{R}^n$.

Observe that

$$\operatorname{diam}(PT) = \sup_{x \in T-T} \left\| Q^\top x \right\|_2 = \sup_{x \in T-T} \max_{z \in S^{m-1}} \langle Q^\top x, z \rangle = \sup_{x \in T-T} \max_{z \in S^{m-1}} \langle Qz, x \rangle.$$

Therefore, we have

$$\begin{aligned}
\mathbb{E} \operatorname{diam}(PT) &= \mathbb{E} \sup_{x \in T-T} \max_{z \in S^{m-1}} \langle Qz, x \rangle \\
&\geq \mathbb{E} \sup_{x \in T-T} \langle Qz_0, x \rangle \\
&= w_s(T-T) \\
&= 2w_s(T).
\end{aligned}$$

Here $z_0 \in S^{m-1}$ is a fixed vector and $Qz_0 \sim \mathrm{Unif}(S^{n-1})$ by Exercise 7.24.

On the other hand, for every fixed $x \in T - T$, we have $Q^\top x \overset{d}{=} \|x\|_2 (\theta_1, \dots, \theta_m)$, where $\theta = (\theta_1, \dots, \theta_n) \sim \mathrm{Unif}(S^{n-1})$. Since $\mathbb{E}\sqrt{\theta_1^2 + \cdots + \theta_m^2} \gtrsim \sqrt{m/n}$, we obtain

$$\mathbb{E} \operatorname{diam}(PT) \geq \sup_{x \in T-T} \mathbb{E} \left\| Q^\top x \right\|_2 \gtrsim \sqrt{\frac{m}{n}} \sup_{x \in T-T} \|x\|_2 = \sqrt{\frac{m}{n}} \operatorname{diam}(T).$$

Combining these two bounds, using $\max\{a, b\} \geq (a+b)/2$, and rescaling completes the proof.

7.27 (Matrix sketching) Suppose we have a large A be an $n \times k$ matrix. Here is how we can reduce it without sacrificing accuracy of the operator norm.

(a) Let P be a projection in $\mathbb{R}^n$ onto a random m-dimensional subspace (picked uniformly in the Grassmanian $G_{n,m}$). Prove that

$$\mathbb{E} \|PA\| \asymp \frac{1}{\sqrt{n}} \|A\|_F + \sqrt{\frac{m}{n}} \|A\|.$$

(b) Let G be an $m \times n$ random matrix with i.i.d. $\mathcal{N}(0,1)$ entries. Prove that

$$\mathbb{E} \|GA\| \asymp \|A\|_F + \sqrt{m} \|A\|.$$

Solution: (a) Let $T = \{Ax : \|x\|_2 \leq 1\}$, then $\operatorname{diam}(T) = 2\|A\|$ and $\operatorname{diam}(PT) = 2\|PA\|$. Let

$g \sim \mathcal{N}(0, I_n)$, we have

$$w(T) = \mathbb{E} \sup_{\|x\|_2 \le 1} \langle g, Ax \rangle = \mathbb{E} \left\| A^\top g \right\|_2 \asymp \|A\|_F .$$

Thus $w_s(T) \asymp w(T)/\sqrt{n} = \|A\|_F / \sqrt{n}$. By Theorem 7.6.1, we have

$$\mathbb{E} \|PA\| = \frac{1}{2} \mathbb{E} \operatorname{diam}(PT) \asymp w_s(T) + \sqrt{\frac{m}{n}} \operatorname{diam}(T) \asymp \frac{1}{\sqrt{n}} \|A\|_F + \sqrt{\frac{m}{n}} \|A\| .$$

We finished the proof.

(b) We can use exactly the same argument as in part (a) and use Exercise 7.25 to get the desired result.

8 Chaining

8.1 (Dudley inequality: a high-probability bound) Modify the chaining argument in Section 8.1 to get the high-probability version of Dudley inequality stated in Remark 8.1.6. To do this, upgrade (8.11) to a high-probability version:

$$\sup_{t \in T} \left(X_{\pi_k(t)} - X_{\pi_{k-1}(t)} \right) \le C \varepsilon_{k-1} \left[\sqrt{\log |T_k|} + z_k \right]$$

with probability at least $1 - 2\exp(-z_k^2)$. Choose the values for z_k to be able to make a union bound over all terms in (8.12).

Solution: We will show that for every $u > 0$, the bound

$$\sup_{t,s \in T} |X_t - X_s| \le CK \left[\int_0^\infty \sqrt{\log \mathcal{N}(T, d, \varepsilon)} d\varepsilon + u \operatorname{diam}(T) \right]$$

holds with probability at least $1 - 2\exp(-u^2)$. First, we transform (8.11) to a high-probability version. For each $t \in T$, we have $\left\| X_{\pi_k(t)} - X_{\pi_{k-1}(t)} \right\|_{\psi_2} \le CK\varepsilon_{k-1}$. Therefore, by the sub-Gaussian tail bound, for a sufficiently large constant C,

$$\mathbb{P} \left\{ \left| X_{\pi_k(t)} - X_{\pi_{k-1}(t)} \right| > CK\varepsilon_{k-1} \left(\sqrt{\log |T_k|} + z_k \right) \right\} \le 2 \exp\left(-2 \log |T_k| - z_k^2 \right) .$$

There are at most $|T_k||T_{k-1}| \le |T_k|^2$ distinct pairs $(\pi_k(t), \pi_{k-1}(t))$. Hence, by a union bound,

$$\mathbb{P}\left\{\sup_{t\in T}\left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| > CK\varepsilon_{k-1}\left(\sqrt{\log|T_k|} + z_k\right)\right\} \le 2|T_k|^2\exp\left(-2\log|T_k| - z_k^2\right) = 2\exp(-z_k^2).$$

Thus,

$$\sup_{t\in T}\left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| \le CK\varepsilon_{k-1}\left(\sqrt{\log|T_k|} + z_k\right)$$

with probability at least $1 - 2\exp(-z_k^2)$.

Let $z_k = u + \sqrt{2\log k} + \sqrt{\log 2}$, then $z_k^2 \ge u^2 + 2\log k + \log 2$ and $\exp(-z_k^2) \le \exp(-u^2)/2k^2$. Choose $\varepsilon_k = 2^{-k}\operatorname{diam}(T)$, then $\pi_0(t) = t_0$ for all $t \in T$ and there exists some k_0 such that $\pi_{k_0}(t) = t$. By a union bound, we have with probability at least $1 - \sum_{k=1}^{k_0} 2\exp(-z_k^2)$,

$$\begin{aligned}
\sup_{t,s}|X_t - X_s| &\le 2\sup_{t\in T}\left|X_t - X_{\pi_0(t)}\right| \\
&\le \sum_{k=1}^{k_0}\sup_{t\in T}\left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| \\
&\le CK\sum_{k=1}^{k_0}\varepsilon_{k-1}\left[\sqrt{\log|T_k|} + z_k\right] \\
&= CK\sum_{k=1}^{k_0}2^{-k}\operatorname{diam}(T)\sqrt{\log|T_k|} + CK\sum_{k=1}^{k_0}2^{-k}\operatorname{diam}(T)z_k.
\end{aligned}$$

For the first term, we have

$$\begin{aligned}
\sum_{k=1}^{k_0}2^{-k}\operatorname{diam}(T)\sqrt{\log|T_k|} &\le \sum_{k=1}^{k_0}2\int_{2^{-k-1}\operatorname{diam}(T)}^{2^{-k}\operatorname{diam}(T)}\sqrt{\log\mathcal{N}(T,d,\varepsilon)}d\varepsilon \\
&\le 2\int_{2^{-k_0-1}\operatorname{diam}(T)}^{2^{-1}\operatorname{diam}(T)}\sqrt{\log\mathcal{N}(T,d,\varepsilon)}d\varepsilon \\
&\le 2\int_0^{\infty}\sqrt{\log\mathcal{N}(T,d,\varepsilon)}d\varepsilon.
\end{aligned}$$

For the second term, we have

$$\begin{aligned}
\sum_{k=1}^{k_0}2^{-k}\operatorname{diam}(T)z_k &\le \sum_{k=1}^{k_0}2^{-k}\operatorname{diam}(T)\left[u + \sqrt{2\log k} + \sqrt{\log 2}\right] \\
&\le \operatorname{diam}(T)\left[u + \sqrt{\log 2} + \sum_{k=1}^{k_0}2^{-k}\sqrt{2\log k}\right] \\
&\le cu\operatorname{diam}(T).
\end{aligned}$$

Also, we have

$$1 - \sum_{k=1}^{k_0} 2\exp(-z_k^2) \geq 1 - \sum_{k=1}^{\infty} \frac{\exp(-u^2)}{k^2} \geq 1 - 2\exp(-u^2).$$

We finish the proof by combining the above results.

8.2 (Dudley for Gaussian processes: a high-probability bound) Give an alternative proof of the result in Remark 8.1.6 for Gaussian processes by combining Dudley inequality with Gaussian concentration (Theorem 7.1.11).

Solution: As $\{X_t\}_{t\in T}$ is a zero-mean Gaussian process, $\{X_t - X_s\}_{t,s\in T}$ is also a Gaussian process with $\sqrt{\mathrm{Var}(X_t - X_s)} \asymp \|X_t - X_s\|_{\psi_2} \leq C_1 K d(t,s)$. Since the index set $T \times T$ contains both (t,s) and (s,t), we have $\sup_{t,s}(X_t - X_s) = \sup_{t,s} |X_t - X_s|$. We apply Theorem 7.1.11 to get

$$\left\| \sup_{t,s\in T} |X_t - X_s| - \mathbb{E} \sup_{t,s\in T} |X_t - X_s| \right\|_{\psi_2} \leq C_2 \sup_{s,t\in T} \sqrt{\mathrm{Var}(X_s - X_t)} \leq C_3 K \operatorname{diam}(T).$$

Therefore, for any $u > 0$, we have

$$\sup_{t,s\in T} |X_t - X_s| \leq \mathbb{E} \sup_{t,s\in T} |X_t - X_s| + C_3 K u \operatorname{diam}(T)$$

with probability at least $1 - 2\exp(-u^2)$. We can finish the proof by noting that

$$\mathbb{E} \sup_{t,s\in T} |X_t - X_s| \leq CK \int_0^\infty \sqrt{\log \mathcal{N}(T,d,\varepsilon)}\, d\varepsilon.$$

8.4 (Dudley inequality can be loose) Let e_k be the standard basis vectors in $\mathbb{R}^n$, and

$$T := \left\{ \frac{e_k}{\sqrt{1+\log k}},\ k = 1, \dots, n \right\}.$$

(a) Show that $w(T) \leq C$.

(b) Show that $\int_0^\infty \sqrt{\log \mathcal{N}(T,d,\varepsilon)}\, d\varepsilon \to \infty$ as $n \to \infty$.

Solution: (a) Let $g \sim N(0, I_n)$, we have

$$w(T) = \mathbb{E} \sup_{t\in T} \langle g, t \rangle = \mathbb{E} \sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}}.$$

For any fixed k,

$$\mathbb{P}\left(\frac{g_k}{\sqrt{1+\log k}} > t\right) \le (ke)^{-t^2/2}.$$

Using the union bound, we have

$$\mathbb{P}\left(\sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}} > t\right) \le \sum_{k=1}^{n} \mathbb{P}\left(\frac{g_k}{\sqrt{1+\log k}} > t\right) \le \sum_{k=1}^{n} (ke)^{-t^2/2}.$$

For $t > 2$, we have

$$\mathbb{P}\left(\sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}} > t\right) \le \sum_{k=1}^{n} (ke)^{-t^2/2} \le e^{-t^2/2} \sum_{k=1}^{\infty} k^{-2} \le C_1 e^{-t^2/2}.$$

For $t < 2$, it trivially holds that

$$\mathbb{P}\left(\sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}} > t\right) \le 1 \le C_2 e^{-t^2/2}.$$

Let $C_3 = \max(C_1, C_2)$,

$$\mathbb{E} \sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}} = \int_0^\infty \mathbb{P}\left(\sup_{k=1,\dots,n} \frac{g_k}{\sqrt{1+\log k}} > t\right) dt \le \int_0^\infty C_3 e^{-t^2/2} dt =: C.$$

We have finished the proof.

(b) Let d denote the Euclidean distance. For distinct $i, j \le m$ and $m \ge 3$, we have

$$d\left(\frac{e_i}{\sqrt{1+\log i}}, \frac{e_j}{\sqrt{1+\log j}}\right)^2 = \frac{1}{1+\log i} + \frac{1}{1+\log j} \ge \frac{2}{1+\log m} \ge \frac{1}{\log m}.$$

Thus, the first m vectors in T form a $1/\sqrt{\log m}$-separated set. Therefore, for $\frac{1}{2\sqrt{\log(m+1)}} \le \varepsilon \le \frac{1}{2\sqrt{\log m}}$, we have $\mathcal{N}(T, d, \varepsilon) \ge m$. It follows that

$$\begin{aligned}
\int_0^\infty \sqrt{\log \mathcal{N}(T, d, \varepsilon)}\, d\varepsilon &\ge \sum_{m=3}^{n-1} \int_{1/(2\sqrt{\log(m+1)})}^{1/(2\sqrt{\log m})} \sqrt{\log m}\, d\varepsilon \\
&= \frac{1}{2} \sum_{m=3}^{n-1} \left(1 - \sqrt{\frac{\log m}{\log(m+1)}}\right) \\
&\ge \frac{1}{16} \sum_{m=3}^{n-1} \frac{1}{m \log m} \\
&\ge \frac{1}{16} \int_3^n \frac{dx}{x \log x}
\end{aligned}$$

$$= \frac{1}{16} \left(\log\log n - \log\log 3 \right).$$

Here we used $\log(1 + 1/m) \geq 1/(m+1)$ and $\log(m+1) \leq 2\log m$ to obtain

$$\begin{aligned} 1 - \sqrt{\frac{\log m}{\log(m+1)}} &= \frac{\log(1+1/m)}{\sqrt{\log(m+1)}\left(\sqrt{\log(m+1)} + \sqrt{\log m}\right)} \\ &\geq \frac{1/(m+1)}{2\log(m+1)} \\ &\geq \frac{1}{8m\log m}. \end{aligned}$$

Thus, the entropy integral tends to infinity as $n \to \infty$.

8.5 (Dudley and Sudakov inequalities are sharp up to log factors) Sudakov inequality (Corollary 7.4.2) and Dudley inequality (Theorem 8.1.8) give these bounds on the Gaussian width $w(T)$ for any bounded set $T \subset \mathbb{R}^n$:

$$s(T) \lesssim w(T) \lesssim d(T),$$

where

$$s(T) = \sup_{\varepsilon > 0} \varepsilon \sqrt{\log \mathcal{N}(T, \varepsilon)} \quad \text{and} \quad d(T) = \int_0^\infty \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon.$$

Prove that these bounds are equivalent up to a logarithmic factor:

$$s(T) \leq d(T) \leq C \log(n)\, s(T).$$

In particular, both Sudakov inequality (Corollary 7.4.2) and Dudley inequality (Theorem 8.1.8) are tight up to a logarithmic factor.

Solution: It is clear that $s(T) \leq d(T)$. Now we prove that $d(T) \preceq \log(n) s(T)$. Since T is bounded, WLOG we can assume that $\operatorname{diam}(T) = 1$. Therefore, we have $\mathcal{N}(T, \varepsilon) = 1$ for any $\varepsilon > 1$. By Exercise 8.3,

$$\int_0^\infty \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon = \int_0^1 \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon \preceq \sum_{k=1}^\infty 2^{-k} \sqrt{\log \mathcal{N}(T, 2^{-k})}.$$

We split the sum into two parts: $2^{-k} \leq n^{-2}$ and $n^{-2} < 2^{-k} \leq 1$. For $2^{-k} \leq n^{-2}$, by the volumetric

argument we have

$$\mathcal{N}(T, 2^{-k}) \le \frac{\mathrm{vol}(T + 2^{-k-1}\mathbb{B}_2^n)}{\mathrm{vol}(2^{-k-1}\mathbb{B}_2^n)} \le \frac{(1 + 2^{-k-1})^n}{2^{-(k+1)n}} \le 2^{(k+2)n},$$

where we use the fact that $\mathrm{diam}(T) = 1$ and $2^{-k} < 1$. Therefore, we have

$$\begin{aligned}
\sum_{k=\lceil 2\log n\rceil}^{\infty} 2^{-k}\sqrt{\log \mathcal{N}(T, 2^{-k})} &\le \sum_{k=\lceil 2\log n\rceil}^{\infty} 2^{-k}\sqrt{(k+2)n\log 2} \\
&\le \frac{1}{n}\sum_{k=\lceil 2\log n\rceil}^{\infty} 2^{-k/2}\sqrt{(k+2)n\log 2} \\
&\le \sqrt{\frac{\log 2}{n}}\sum_{k=\lceil 2\log n\rceil}^{\infty} 2^{-k/2}\sqrt{k+2} \\
&\preceq \sqrt{\frac{1}{n}},
\end{aligned}$$

where the second inequality uses the fact that $2^{-k/2} \le 1/n$ for $k \ge 2\log n$. For the sum where $n^{-2} < 2^{-k} \le 1$, there are at most $C\log n$ terms, and for each term we have

$$2^{-k}\sqrt{\log \mathcal{N}(T, 2^{-k})} \le \sup_{\varepsilon > 0} \varepsilon\sqrt{\log \mathcal{N}(T, \varepsilon)} \coloneqq s(T).$$

Therefore, we can conclude

$$\int_0^{\infty} \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon \preceq \log(n) s(T).$$

8.6 (Dudley inequality with refined limits) We know from Remark 8.1.7 that the upper limit in the Dudley integral (8.17) can be set to the diameter of T. It would be useful to improve the lower limit, too, since the integral might blow up near 0. Show that the Gaussian width of any bounded set $T \subset \mathbb{R}^n$ satisfies

$$w(T) \le C\int_a^b \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon \quad \text{where} \quad a = \frac{cw(T)}{\sqrt{n}}, \qquad b = \mathrm{diam}(T).$$

Solution: Let the dyadic scale $\varepsilon_k = 2^{-k}$. If $w(T) = 0$, the result is trivial, so assume that $w(T) > 0$. We choose $\kappa \in \mathbb{Z}$ such that $\mathrm{diam}(T) \le 2^{-\kappa} < 2\,\mathrm{diam}(T)$, so that $\mathcal{N}(T, 2^{-\kappa}) = 1$, and let $K \in \mathbb{Z}$ be the smallest integer such that $2^{-K} \le \frac{w(T)}{8\sqrt{n}}$. For each ε_k, let T_k be a minimal ε_k-net of T and write $\pi_k(t)$ as the closest point in T_k to $t \in T$. Suppose $g \sim N(0, I_n)$, and let $X_t = \langle g, t\rangle$ for $t \in T$. Then X_t is a Gaussian process with subgaussian increments, that is,

$\|X_t - X_s\|_{\psi_2} \asymp \sqrt{\mathrm{Var}(X_t - X_s)} = \|t - s\|_2$.

Therefore, we have

$$\begin{aligned} w(T) &= \mathbb{E} \sup_{t \in T} X_t \\ &= \mathbb{E} \sup_{t \in T} (X_t - X_{\pi_\kappa(t)}) \\ &= \mathbb{E} \sup_{t \in T} \sum_{k=\kappa+1}^{K} (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) + \mathbb{E} \sup_{t \in T} (X_t - X_{\pi_K(t)}) \\ &\leq \sum_{k=\kappa+1}^{K} \mathbb{E} \sup_{t \in T} (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) + \mathbb{E} \sup_{t \in T} (X_t - X_{\pi_K(t)}). \end{aligned}$$

By similar techniques as in the proof of Theorem 8.1.4,

$$\mathbb{E} \sup_{t \in T} (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) \preceq \varepsilon_{k-1} \sqrt{\log \mathcal{N}(T, \varepsilon_k)}.$$

This further implies that

$$\sum_{k=\kappa+1}^{K} \mathbb{E} \sup_{t \in T} (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) \preceq \sum_{k=\kappa+1}^{K} 2^{-k+1} \sqrt{\log \mathcal{N}(T, 2^{-k})} \preceq \int_{2^{-K-1}}^{2^{-\kappa-1}} \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon.$$

Now we bound the last term $\mathbb{E} \sup_{t \in T} (X_t - X_{\pi_K(t)})$. By definition of X_t,

$$\begin{aligned} \mathbb{E} \sup_{t \in T} (X_t - X_{\pi_K(t)}) &= \mathbb{E} \sup_{t \in T} \langle g, t - \pi_K(t) \rangle \\ &\leq \sqrt{\mathbb{E} \sup_{t \in T} \langle g, t - \pi_K(t) \rangle^2} \\ &\leq \sqrt{\mathbb{E} \sup_{t \in T} \|g\|^2 \, \|t - \pi_K(t)\|_2^2} \\ &\leq \sqrt{n} \cdot 2^{-K} \\ &\leq \sqrt{n} \cdot \frac{w(T)}{8\sqrt{n}} \\ &= \frac{w(T)}{8}. \end{aligned}$$

Therefore, we can conclude that there exsists some constant C such that

$$w(T) \leq C \int_{2^{-K-1}}^{2^{-\kappa-1}} \sqrt{\log \mathcal{N}(T, \varepsilon)}\, d\varepsilon + \frac{w(T)}{8}.$$

Absorbing $w(T)/8$ into the left-hand side and noting that $2^{-K-1} \asymp w(T)/\sqrt{n}$ and $2^{-\kappa-1} \le \operatorname{diam}(T)$, we obtain the claim by choosing $c > 0$ sufficiently small.

8.7 (Subexponential Dudley inequality) Suppose that the increments of a mean-zero random process $(X_t)_{t\in T}$ are not subgaussian as in (8.1) but subexponential:

$$\|X_t - X_s\|_{\psi_1} \le K d(t,s) \qquad \text{for all } t, s \in T.$$

Prove the following version of Dudley inequality (Theorem 8.1.3):

$$\mathbb{E} \sup_{t\in T} X_t \le CK \int_0^\infty \log \mathcal{N}(T, d, \varepsilon)\, d\varepsilon.$$

Solution: Let the dyadic scale $\varepsilon_k = 2^{-k}$. For each ε_k, let T_k be a minimal ε_k-net of T and write $\pi_k(t)$ as the closest point in T_k to $t \in T$. By Remark 7.2.1, WLOG we assume that T is finite, so there exist some $\kappa, K \in \mathbb{Z}$ such that $\pi_\kappa(t) = t_0$ and $\pi_K(t) = t$ for all $t \in T$.

Following the chaining argument, we have

$$\begin{aligned}
\mathbb{E} \sup_t X_t &= \mathbb{E} \sup_t (X_t - X_{t_0}) \\
&= \mathbb{E} \sup_t \sum_{k=\kappa+1}^{K} (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) \\
&\le \sum_{k=\kappa+1}^{K} \mathbb{E} \sup_t (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}).
\end{aligned}$$

For each k, we have $\left\|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right\|_{\psi_1} \le K d(\pi_k(t), \pi_{k-1}(t)) \le 2K\varepsilon_{k-1}$. And the supremum is taken over at most $|T_k||T_{k-1}| \le |T_k|^2$ distinct pairs $(\pi_k(t), \pi_{k-1}(t))$. By Exercise 2.44(b), there exists some constant C such that

$$\mathbb{E} \max_{t\in T} \left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| \le CK\varepsilon_{k-1} \log |T_k|.$$

Therefore,

$$\sum_{k=\kappa+1}^{K} \mathbb{E} \sup_t (X_{\pi_k(t)} - X_{\pi_{k-1}(t)}) \le \sum_{k=\kappa+1}^{K} CK\varepsilon_{k-1} \log |T_k|$$

$$
\begin{aligned}
&\le CK \sum_{k=\kappa+1}^{K} 2^{-k} \log \mathcal{N}(T, 2^{-k}) \\
&= 2CK \sum_{k=\kappa+1}^{K} \int_{2^{-k-1}}^{2^{-k}} \log \mathcal{N}(T, 2^{-k})\, d\varepsilon \\
&\le 2CK \sum_{k=\kappa+1}^{K} \int_{2^{-k-1}}^{2^{-k}} \log \mathcal{N}(T, \varepsilon)\, d\varepsilon \\
&\le 2CK \int_{2^{-K-1}}^{2^{-\kappa-1}} \log \mathcal{N}(T, \varepsilon)\, d\varepsilon \\
&\le 2CK \int_{0}^{\infty} \log \mathcal{N}(T, \varepsilon)\, d\varepsilon.
\end{aligned}
$$

We finish the proof.

8.8 (Local Dudley inequality) Let $(X_t)_{t\in T}$ be a random process on a metric space (T, d) with subgaussian increments as in (8.1). Let $\delta > 0$. Prove that

$$
\mathbb{E} \sup_{\substack{s,t\in T \\ d(s,t)\le\delta}} |X_s - X_t| \le CK \int_0^{\delta} \sqrt{\log \mathcal{N}(T, d, \varepsilon)}\, d\varepsilon.
$$

Solution: Define the index set $V = \{(s,t) \in T \times T : d(s,t) \le \delta\}$ and let $Y_{(s,t)} = (X_s - X_t)$ for each $(s,t) \in T \times T$. Define the metric $\rho((s_1,t_1),(s_2,t_2)) = \frac{1}{2} \left\|Y_{(s_1,t_1)} - Y_{(s_2,t_2)}\right\|_{\psi_2} / K$ on $T \times T$. It is easy to check that $\operatorname{diam}(V) \le \delta$ with respect to ρ.

By definition of ρ, we also have $\left\|Y_{(s_1,t_1)} - Y_{(s_2,t_2)}\right\|_{\psi_2} \le 2K\rho((s_1,t_1),(s_2,t_2))$. Also, suppose $\{t_1, t_2, \ldots, t_k\}$ are an ϵ-net of T with respect to d, then $\{(t_i, t_j) : 1 \le i, j \le k\}$ is an external ϵ-net of V with respect to ρ. Concretely, for any $(s,t) \in V$, there exist t_i, t_j such that $d(s,t_i) \le \epsilon$ and $d(t,t_j) \le \epsilon$, and we have

$$
\begin{aligned}
\rho((s,t),(t_i,t_j)) &= \frac{1}{2K} \left\|Y_{(s,t)} - Y_{(t_i,t_j)}\right\|_{\psi_2} \\
&\le \frac{1}{2K} \left\|X_s - X_{t_i}\right\|_{\psi_2} + \frac{1}{2K} \left\|X_t - X_{t_j}\right\|_{\psi_2} \\
&\le \frac{1}{2K} K d(s,t_i) + \frac{1}{2K} K d(t,t_j) \\
&\le \epsilon.
\end{aligned}
$$

Let $\overline{\mathcal{N}}(V, \rho, \epsilon)$ denote the external covering number of V, where the centers of the covering balls

are allowed to lie in $T \times T$ rather than in V. Thus $\overline{\mathcal{N}}(V, \rho, \epsilon) \le \mathcal{N}(T, d, \epsilon)^2$. By Exercise 4.25,

$$\mathcal{N}(V, \rho, 2\epsilon) \le \overline{\mathcal{N}}(V, \rho, \epsilon) \le \mathcal{N}(T, d, \epsilon)^2.$$

Since the process X_t is mean-zero, the process $Y_{(s,t)}$ is also mean-zero. Moreover, V contains (s,t) if and only if it contains (t,s), and $Y_{(t,s)} = -Y_{(s,t)}$. Now we apply Theorem 8.1.3 to the process $\{Y_{(s,t)}\}_{(s,t)\in V}$ with respect to the metric ρ:

$$\begin{aligned}
\mathbb{E} \sup_{(s,t)\in V} \left|Y_{(s,t)}\right| &= \mathbb{E} \sup_{(s,t)\in V} Y_{(s,t)} \\
&\le CK \int_0^{\operatorname{diam}(V)} \sqrt{\log \mathcal{N}(V, \rho, \varepsilon)}\, d\varepsilon \\
&\le CK \int_0^{\delta} \sqrt{\log \mathcal{N}(T, d, \varepsilon/2)}\, d\varepsilon \\
&= 2CK \int_0^{\delta/2} \sqrt{\log \mathcal{N}(T, d, u)}\, du \\
&\le 2CK \int_0^{\delta} \sqrt{\log \mathcal{N}(T, d, u)}\, du,
\end{aligned}$$

which is exactly the desired result.

8.9 (Covering numbers of Lipschitz functions) For the class of functions

$$\mathcal{F} := \left\{ f : [0,1] \to [0,1],\ \|f\|_{\mathrm{Lip}} \le 1 \right\},$$

show that

$$\mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon) \le e^{C/\varepsilon} \qquad \text{for any } \varepsilon \in (0,1).$$

Solution: Fix $\varepsilon \in (0,1)$, let $m = \lceil 3/\varepsilon \rceil$ and $h = 1/m$, and consider the grid points $x_i = ih$ for $i = 0, 1, \ldots, m$. For each $f \in \mathcal{F}$, let $q_i(f)$ be a closest point to $f(x_i)$ in the grid $\{0, h, 2h, \ldots, 1\}$, and write $q(f) = (q_0(f), \ldots, q_m(f))$.

Suppose that $q(f) = q(g)$ for two functions $f, g \in \mathcal{F}$. Then $|f(x_i) - g(x_i)| \le h$ for every i. Thus, for any $x \in [x_i, x_{i+1}]$, the Lipschitz property gives

$$|f(x) - g(x)| \le |f(x) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(x)| \le 3h \le \varepsilon.$$

It remains to count the possible vectors $q(f)$. There are $m+1$ choices for $q_0(f)$. Moreover, the

Lipschitz property gives $|q_{i+1}(f) - q_i(f)| \le h/2 + h + h/2 = 2h$, so after $q_i(f)$ is fixed, there are at most five choices for $q_{i+1}(f)$. For each vector $q(f)$ that occurs, choose one corresponding function in $\mathcal{F}$. These functions form an ε-net of $\mathcal{F}$, and therefore

$$\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon) \le \log(m+1) + m \log 5 \le \frac{C}{\varepsilon}.$$

Exponentiating both sides completes the proof.

8.10 (Lipschitz law of large numbers in high dimensions) Let's generalize Theorem 8.2.3 to high dimensions. Consider functions on the unit cube that are L-Lipschitz with respect to the $\|\cdot\|_\infty$ metric in $\mathbb{R}^n$:

$$\mathcal{F} := \left\{ f : [0,1]^d \to \mathbb{R},\ \|f\|_{\text{Lip}} \le L \right\}.$$

Let $X, X_1, X_2, \dots, X_n$ be i.i.d. random variables taking values in $[0,1]$. Show that

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbb{E} f(X) \right| \le CL \begin{cases} (\log n) n^{-1/2}, & d = 2, \\ n^{-1/d}, & d \ge 3. \end{cases}$$

To prove this, assume $L = 1$ and follow these steps:

(a) Extend the bound on the covering numbers (8.9) to higher dimensions:

$$\mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon) \le e^{C/\varepsilon^d}.$$

(b) If you just plug this into (8.25), the integral will diverge because of the singularity at zero. To fix this, refine (8.25) to

$$\mathbb{E} \sup_{f \in \mathcal{F}} |X_f| \lesssim \delta + \frac{1}{\sqrt{n}} \int_\delta^1 \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon)}\, d\varepsilon \qquad \text{for any } \delta \in [0,1].$$

Solution: (a) Since the empirical process is unchanged when a constant is added to f, we may normalize every function by $f(0) = 0$ and work with the class $\mathcal{F}_0 = \{f \in \mathcal{F} : f(0) = 0\}$. Notice that $\|f\|_{L^\infty} \le 1$ for every $f \in \mathcal{F}_0$. The covering number bound in part (a) is understood for this normalized class.

Fix $\varepsilon \in (0,1)$, let $m = \lceil 3/\varepsilon \rceil$ and $h = 1/m$, and consider the grid $G = \{0, h, 2h, \dots, 1\}^d$. For each $f \in \mathcal{F}_0$ and $v \in G$, let $q_v(f)$ be a closest point to $f(v)$ in the grid $\{-1, -1+h, \dots, 1\}$.

Suppose that $q_v(f) = q_v(g)$ for every $v \in G$. For any $x \in [0,1]^d$, choose $v \in G$ such that $\|x - v\|_\infty \le h$. Then

$$|f(x) - g(x)| \le |f(x) - f(v)| + |f(v) - g(v)| + |g(v) - g(x)| \le 3h \le \varepsilon.$$

To count the possible collections $(q_v(f))_{v \in G}$, choose a spanning tree of the grid rooted at the origin. If v is connected to its parent v', then $\|v - v'\|_\infty = h$ and thus $|q_v(f) - q_{v'}(f)| \le h/2 + h + h/2 = 2h$. Since $q_0(f) = 0$, each remaining quantized value has at most five choices once its parent value is fixed. For each collection that occurs, choose one corresponding function in $\mathcal{F}_0$. These functions form an ε-net, and

$$\log \mathcal{N}(\mathcal{F}_0, \|\cdot\|_{L^\infty}, \varepsilon) \le (|G| - 1) \log 5 \le (m+1)^d \log 5 \le \frac{C}{\varepsilon^d},$$

where C depends only on d. This completes the proof of part (a).

(b) The remedy is simple. Since for any $f \in \mathcal{F}$, $\|f\|_{L^\infty} \le 1$, we have that for any $\varepsilon \ge 1$, $\mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon) = 1$. Since $\mathcal{F}$ is symmetric and $X_{-f} = -X_f$, we also have $\sup_{f \in \mathcal{F}} |X_f| = \sup_{f \in \mathcal{F}} X_f$. Fix $\delta \in (0,1)$, and choose a sufficiently large $K \in \mathbb{N}$ such that $2^K \delta \ge 1$. Let $\varepsilon_k = 2^{K-k+1}\delta$ for $0 \le k \le K$. Then $\varepsilon_K = 2\delta$ and $\varepsilon_{k-1} = 2\varepsilon_k$. Let $\mathcal{F}_0 = \{0\}$, and for each $1 \le k \le K$, let $\mathcal{F}_k$ be a minimal ε_k-net of $\mathcal{F}$. Write $\pi_k(f)$ as the closest point in $\mathcal{F}_k$ to $f \in \mathcal{F}$. Then we have

$$\mathbb{E} \sup_{f \in \mathcal{F}} X_f \le \mathbb{E} \sup_{f \in \mathcal{F}} (X_f - X_{\pi_K(f)}) + \sum_{k=1}^{K} \mathbb{E} \sup_{f \in \mathcal{F}} (X_{\pi_k(f)} - X_{\pi_{k-1}(f)}).$$

Since $\|f - \pi_K(f)\|_{L^\infty} \le 2\delta$, the first term on the right-hand side is bounded by 4δ.

As in the proof of Theorem 8.2.3, $\|X_f - X_g\|_{\psi_2} \le C \|f - g\|_{L^\infty} / \sqrt{n}$. Moreover, the supremum at level k is taken over at most $|\mathcal{F}_k||\mathcal{F}_{k-1}| \le |\mathcal{F}_k|^2$ distinct pairs, and $\|\pi_k(f) - \pi_{k-1}(f)\|_{L^\infty} \le \varepsilon_k + \varepsilon_{k-1}$. The maximal inequality therefore gives

$$\mathbb{E} \sup_{f \in \mathcal{F}} (X_{\pi_k(f)} - X_{\pi_{k-1}(f)}) \le \frac{C\varepsilon_{k-1}}{\sqrt{n}} \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon_k)}.$$

Since the covering numbers are nonincreasing in ε, summing these bounds yields

$$\sum_{k=1}^{K} \varepsilon_{k-1} \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon_k)} \le 4 \sum_{k=1}^{K} \int_{\varepsilon_k/2}^{\varepsilon_k} \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon)}\, d\varepsilon$$

$$= 4 \int_\delta^{\varepsilon_1} \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon)}\, d\varepsilon$$
$$= 4 \int_\delta^{1} \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_{L^\infty}, \varepsilon)}\, d\varepsilon.$$

Combining the last three displays proves the desired refined bound.

8.11 (Symmetrization for empirical processes) Modify the proof of Symmetrization Lemma 6.3.2 to obtain the following version of symmetrization. Let $\mathcal{F}$ be a class of Boolean functions on some domain Ω, and let $X, X_1, X_2, \ldots, X_n$ be independent random points in Ω with common distribution. Prove that

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbb{E} f(X) \right| \le 2\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(X_i) \right|,$$

where $\varepsilon_1, \varepsilon_2, \ldots$ are independent Rademacher random variables (which are also independent of $X_1, X_2, \ldots$).

Solution: For each i, define the random vector $Y_i = (f(X_i))_{f \in \mathcal{F}}$ in $\ell_\infty(\mathcal{F})$. Applying Exercise 6.19(a) to $Y_1, \ldots, Y_n$ and dividing by n, we obtain

$$\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n f(X_i) - \mathbb{E} f(X) \right| \le 2\mathbb{E} \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(X_i) \right|,$$

as desired.

8.17 (VC dimension of half-spaces) Prove the result in Example 8.3.5: if $\mathcal{F}$ is the class of indicators of all half-planes in $\mathbb{R}^n$, then

$$\mathrm{vc}(\mathcal{F}) = n + 1.$$

Solution: We prove by induction. Let $\mathcal{F}_n$ denote the class of indicators of all half-spaces in $\mathbb{R}^n$. It is easy to verify $\mathrm{vc}(\mathcal{F}_1) = 2$, and Example 8.3.3 gives $\mathrm{vc}(\mathcal{F}_2) = 3$. Suppose $\mathrm{vc}(\mathcal{F}_n) = n + 1$ for all $n \le m$.

Choose a set $\Lambda \subseteq \mathbb{R}^m$ of $m + 1$ points shattered by $\mathcal{F}_m$. Any labeling of $(\Lambda \times \{0\}) \cup \{e_{m+1}\}$ is realized by a half-space $\{(x, t) : \langle a, x \rangle + ct \le b\}$: choose a, b to realize the labels on Λ, then choose $c = b - 1$ or $c = b + 1$ to include or exclude e_{m+1}. Thus, $\mathrm{vc}(\mathcal{F}_{m+1}) \ge m + 2$.

For the reverse bound, take any $m + 3$ points $x_1, \ldots, x_{m+3}$ in $\mathbb{R}^{m+1}$. Their lifts $(x_i, 1)$ in $\mathbb{R}^{m+2}$ are linearly dependent, so there are coefficients α_i, not all zero, such that $\sum_{i=1}^{m+3} \alpha_i x_i = 0$ and

$\sum_{i=1}^{m+3} \alpha_i = 0$. There are both positive and negative coefficients. If these points were shattered, we could choose a, b such that $\langle a, x_i \rangle \le b$ for $\alpha_i > 0$ and $\langle a, x_i \rangle > b$ for $\alpha_i < 0$. But then

$$0 = \sum_{i=1}^{m+3} \alpha_i (\langle a, x_i \rangle - b) < 0,$$

a contradiction. Hence $\mathrm{vc}(\mathcal{F}_{m+1}) \le m + 2$, completing the induction.

8.18 (VC dimension vs. algebraic dimension) Show that for any finite-dimensional class of Boolean functions $\mathcal{F}$ on a domain Ω, we have

$$\mathrm{vc}(\mathcal{F}) \le \dim(\mathcal{F}),$$

where $\dim(\mathcal{F})$ denotes the linear algebraic dimension of $\mathcal{F}$, i.e. the maximal number of linearly independent functions in $\mathcal{F}$.

Solution: Let $n \coloneqq \dim(\mathcal{F})$. It suffices to show that any set with $n + 1$ elements cannot be shattered by $\mathcal{F}$. Suppose $\{x_1, \dots, x_{n+1}\}$ can be shattered by $\mathcal{F}$. Then there exists $f_1, \dots, f_{n+1}$ such that $f_i(x_j) = \delta_{ij}$, that is, when $f_i(x_j) = 1$ when $i = j$ and $f_i(x_j) = 0$ when $i \ne j$. Then it is easy to verify that $f_1, \dots, f_{n+1}$ are linearly independent, which contradicts $\dim(\mathcal{F}) = \backslash$.

8.19 (Sharpness of Pajor and Sauer–Shelah lemmas) Show that both Pajor Lemma 8.3.7 and Sauer–Shelah Lemma 8.3.9 are sharp by considering the set $\mathcal{F}$ of binary strings of length n with at most d ones (this set is often called a Hamming ball).

Solution: Let $\Omega = \{1, \dots, n\}$ and let $0 \le d \le n$ be an integer. We identify each binary string with a Boolean function $f : \Omega \to \{0, 1\}$. Thus, $\mathcal{F}$ consists of the functions with at most d ones, and choosing their positions gives $|\mathcal{F}| = \sum_{k=0}^{d} \binom{n}{k}$.

We claim that a subset $\Lambda \subseteq \Omega$ is shattered by $\mathcal{F}$ if and only if $|\Lambda| \le d$. Indeed, if $|\Lambda| \le d$, any Boolean function on Λ can be extended by zero outside Λ. This extension has at most d ones and hence belongs to $\mathcal{F}$. Conversely, if $|\Lambda| > d$, no function in $\mathcal{F}$ can equal one at every point of Λ, so Λ cannot be shattered.

Thus, including the empty set, the number of shattered subsets is exactly

$$|\{\Lambda \subseteq \Omega : \Lambda \text{ is shattered by } \mathcal{F}\}| = \sum_{k=0}^{d} \binom{n}{k} = |\mathcal{F}|.$$

Therefore, equality holds in Pajor Lemma. Moreover, the claim implies that $\mathrm{vc}(\mathcal{F}) = d$, and the formula for $|\mathcal{F}|$ shows that equality also holds in the first inequality of Sauer–Shelah Lemma. This proves that both lemmas are sharp.

8.21 (VC stability) In Proposition 8.3.11, we observed how the VC dimension is stable when we combined two function classes by taking pointwise minima or maxima. Let us generalize this observation for any number of classes, and for any way of combining them. Let $\mathcal{F}_1, \dots, \mathcal{F}_k$ be classes of Boolean functions on the same domain. Fix any function $\phi : \{0,1\}^k \to \{0,1\}$, which we can think of as a Boolean formula. Consider the class

$$\mathcal{F} = \{\phi(f_1, \dots, f_k) : f_1 \in \mathcal{F}_1, \dots, f_k \in \mathcal{F}_k\}.$$

Prove that if $\mathrm{vc}(\mathcal{F}_i) \le d$ for every i, then

$$\mathrm{vc}(\mathcal{F}) \le Cdk \log k,$$

where C is an absolute constant.

Solution: Assume $k \ge 2$ and $d \ge 1$, and let $n \ge d$ be the size of any set shattered by $\mathcal{F}$. Since $\phi(f_1, \dots, f_k)$ is determined by its k inputs, Sauer–Shelah Lemma gives

$$2^n = \Pi_{\mathcal{F}}(n) \le \prod_{i=1}^{k} \Pi_{\mathcal{F}_i}(n) \le \Big(\frac{en}{d}\Big)^{kd}.$$

Set $x = n/(kd)$. Taking logarithms gives $x \log 2 \le \log(ek) + \log x$. If $x \ge 4$, then $\log x \le x \log 2/2$, so $x \le 2\log(ek)/\log 2 \le C \log k$. If $x < 4$, we also have $x \le C \log k$ since $k \ge 2$. Thus $n \le Cdk \log k$, proving the result. The cases $n < d$ and $d = 0$ are immediate.

8.22 (VC dimension of the union)

(a) Let $\mathcal{F}$ and $\mathcal{G}$ be classes of Boolean functions on the same domain. Prove that

$$\mathrm{vc}(\mathcal{F} \cup \mathcal{G}) \le \mathrm{vc}(\mathcal{F}) + \mathrm{vc}(\mathcal{G}) + 1.$$

(b) Give an example where $\mathcal{F}$ and $\mathcal{G}$ have positive VC dimensions, and

$$\mathrm{vc}(\mathcal{F} \cup \mathcal{G}) = \mathrm{vc}(\mathcal{F}) + \mathrm{vc}(\mathcal{G}) + 1.$$

Solution: (a) Define $d_1 \coloneqq \mathrm{vc}(\mathcal{F})$ and $d_2 \coloneqq \mathrm{vc}(\mathcal{G})$. Suppose there is a set Λ of size $n \geq d_1 + d_2 + 2$ that is shattered by $\mathcal{F} \cup \mathcal{G}$. Then we have

$$2^n = \Pi_{\mathcal{F}\cup\mathcal{G}}(n) \leq \Pi_{\mathcal{F}}(n) + \Pi_{\mathcal{G}}(n) \leq \sum_{k=0}^{d_1} \binom{n}{k} + \sum_{k=0}^{d_2} \binom{n}{k}.$$

On the other hand, by symmetry, $\sum_{k=0}^{d_2} \binom{n}{k} = \sum_{k=n-d_2}^{n} \binom{n}{k}$. Since $n - d_2 \geq d_1 + 2$, the two sums on the right-hand side cover disjoint ranges and omit the terms with $d_1 < k < n - d_2$. So the right-hand side is strictly less than $\sum_{k=0}^{n} \binom{n}{k} = 2^n$, a contradiction.

(b) We consider two set of binary strings. Let

$$\mathcal{F} = \{011, 111, 100, 110\}, \quad \mathcal{G} = \{010, 001, 101, 000\}.$$

It is easy to verify that $\mathrm{vc}(\mathcal{F}) = \mathrm{vc}(\mathcal{G}) = 1$, and $\mathrm{vc}(\mathcal{F} \cup \mathcal{G}) = 3$.

8.23 Theorem 8.3.13 is stated for $\varepsilon \in (0, 1)$. What bound holds for larger ε?

Solution: Since $\mathcal{F}$ consists of Boolean functions and μ is a probability measure, for any $f, g \in \mathcal{F}$,

$$\|f - g\|^2_{L^2(\mu)} = \int_\Omega |f - g|^2 \, d\mu \leq 1.$$

Thus, for $\varepsilon \geq 1$, any singleton $\{f_0\} \subseteq \mathcal{F}$ is an ε-net, and hence $\mathcal{N}(\mathcal{F}, L^2(\mu), \varepsilon) = 1$.

8.24 (A simpler, weaker VC law of large numbers) Use Sauer–Shelah Lemma directly, instead of Pajor Lemma, to prove a weaker version of Theorem 8.3.15, with $C\sqrt{(d/n)\log(en/d)}$ in the right hand side, where $d = \mathrm{vc}(\mathcal{F})$.

Solution: By symmetrization,

$$\mathbb{E} \sup_{f\in\mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}f(X) \right| \leq 2\mathbb{E} \sup_{f\in\mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(X_i) \right|.$$

Condition on $X_1, \ldots, X_n$ and set $\mathcal{F}_X = \{(f(X_1), \ldots, f(X_n)) : f \in \mathcal{F}\}$. Sauer–Shelah Lemma gives

$|\mathcal{F}_X| \le (en/d)^d$. For each $v \in \mathcal{F}_X \subseteq \{0,1\}^n$, we have $\left\|n^{-1}\sum_{i=1}^n \varepsilon_i v_i\right\|_{\psi_2} \le C/\sqrt{n}$. Thus, the maximal inequality (Proposition 2.7.6) yields

$$\mathbb{E}_\varepsilon \max_{v\in\mathcal{F}_X}\left|\frac{1}{n}\sum_{i=1}^n \varepsilon_i v_i\right| \le C\sqrt{\frac{\log(2|\mathcal{F}_X|)}{n}} \le C\sqrt{\frac{d}{n}\log\frac{en}{d}}.$$

Taking expectation over the sample completes the proof.

8.25 (Learning 1-dimensional marginals) Learning high-dimensional distributions from data is hard. But we can learn all 1-dimensional marginals from just $O(n)$ samples! Let $X, X_1, \dots, X_m$ be i.i.d. random vectors in $\mathbb{R}^n$. Show that the CDFs of all 1-dimensional marginals $\langle X, u\rangle$ can be estimated uniformly from the data, using the empirical CDFs:

$$\mathbb{E}\sup_{u\in S^{n-1},\, t\in\mathbb{R}}\left|\frac{1}{m}\sum_{i=1}^m \mathbb{1}_{\{\langle X_i,u\rangle\le t\}} - \mathbb{P}\{\langle X,u\rangle \le t\}\right| \lesssim \sqrt{\frac{n}{m}}.$$

Solution: We consider the function class

$$\mathcal{F} = \left\{f_{u,t} :\ \mathbb{R}^n \to \{0,1\},\ x \mapsto \mathbb{1}_{\{\langle X_i,u\rangle\le t\}} \mid u\in\mathbb{R}^n, t\in\mathbb{R}\right\}.$$

By Exercise 8.17, $\mathrm{vc}(\mathcal{F}) = n+1$. Therefore, we can get the conclusion by applying the VC law of large numbers (Theorem 8.3.15).

8.26 (One-bit quantization) The simplest way to quantize a unit vector $u \in \mathbb{R}^n$ is to take the sign of each coordinate: set $\Psi(u) = \mathrm{sign}(u)$, where the sign[2] function is applied to each coordinate of u. But this can be quite inaccurate—very close vectors u and v can be sent to very different outputs. (Can you see why?) Let's try something smarter.

(a) Let A be an $m \times n$ random matrix with independent standard normal entries. Consider the map $\Phi : S^{n-1} \to \{-1,1\}^m$ given by

$$\Phi(u) = \mathrm{sign}(Au),$$

[2] Let's disallow zero values, redefining the sign as $\mathrm{sign}(t) = 1$ if $t \ge 0$ and $\mathrm{sign}(t) = -1$ if $t < 0$.

where the sign function is applied to each coordinate of Au. Check that

$$\mathbb{E}\frac{1}{m}d(\Phi(u),\Phi(v)) = \frac{1}{\pi}\rho(u,v) \qquad \text{for all } u,v \in S^{n-1}.$$

Here $d(\cdot,\cdot)$ denotes the Hamming distance on the binary cube $\{-1,1\}^m$ and $\rho(\cdot,\cdot)$ denotes the geodesic distance[3] on the sphere S^{n-1}.

(b) Use the VC law of large numbers (Theorem 8.3.15) to show that the following event occurs with probability at least 0.99:

$$\left|\frac{1}{m}d(\Phi(u),\Phi(v)) - \frac{1}{\pi}\rho(u,v)\right| \le C\sqrt{\frac{n}{m}} \qquad \text{for all } u,v \in S^{n-1}.$$

Solution: (a) We use $X_1,\dots,X_m$ to denote the rows of A. Then

$$\Phi(u) = (\operatorname{sign}(\langle X_1,u\rangle),\dots,\operatorname{sign}(\langle X_m,u\rangle))^\top.$$

Note that

$$d(\Phi(u),\Phi(v)) = \frac{m - \langle\Phi(u),\Phi(v)\rangle}{2},$$

we have

$$\begin{aligned}
\mathbb{E}\frac{1}{m}d(\Phi(u),\Phi(v)) &= \frac{1}{2m}\mathbb{E}\left(m - \langle\Phi(u),\Phi(v)\rangle\right)\\
&= \frac{1}{2m}\mathbb{E}\left(m - \sum_{i=1}^m \operatorname{sign}(\langle X_i,u\rangle)\operatorname{sign}(\langle X_i,v\rangle)\right)\\
&= \frac{1}{2}\left(1 - \mathbb{E}\operatorname{sign}(\langle X_1,u\rangle)\operatorname{sign}(\langle X_1,v\rangle)\right)\\
&= \mathbb{P}\{\operatorname{sign}(\langle X_1,u\rangle) \ne \operatorname{sign}(\langle X_1,v\rangle)\}.
\end{aligned}$$

Let $P = \operatorname{span}\{u,v\}$ and let $C = P \cap S^{n-1}$ be the great circle through u and v. The shorter arc joining u and v has length $\theta = \rho(u,v)$. By rotational invariance of X_1, the line $X_1^\perp \cap P$ has a uniform angular direction. The signs of $\langle X_1,u\rangle$ and $\langle X_1,v\rangle$ differ precisely when this line meets the shorter arc joining u and v, and hence also its antipodal arc. These two arcs have total length

[3]For Hamming distance, recall Definition 4.2.14. The geodesic distance on the sphere is the length of the smallest arc that connects the two points.

2θ, while the great circle has circumference 2π. Therefore,

$$\mathbb{E}\frac{1}{m}d(\Phi(u),\Phi(v)) = \mathbb{P}\{\text{sign}(\langle X_1, u\rangle) \neq \text{sign}(\langle X_1, v\rangle)\} = \frac{2\theta}{2\pi} = \frac{\rho(u,v)}{\pi}.$$

(b) Following the same argument as part (a),

$$\begin{aligned}&\left|\frac{1}{m}d(\Phi(u),\Phi(v)) - \frac{1}{\pi}\rho(u,v)\right| \\ &= \left|\frac{1}{m}\sum_{i=1}^{m} \mathbb{1}\left\{\text{sign}(\langle X_i, u\rangle) \neq \text{sign}(\langle X_i, v\rangle)\right\} - \mathbb{E}\mathbb{1}\left\{\text{sign}(\langle X_i, u\rangle) \neq \text{sign}(\langle X_i, v\rangle)\right\}\right|.\end{aligned}$$

Define the function class

$$\mathcal{F} = \{\mathbb{1}\{\text{sign}(\langle x, u\rangle) \neq \text{sign}(\langle x, v\rangle)\} \mid u, v \in \mathbb{S}^{n-1}\}.$$

$\mathcal{F}$ is contained in the class $\{\phi(f_1, f_2) \mid f_1, f_2 \in \mathcal{F}'\}$, where $\phi(x,y) = \mathbb{1}\{x \neq y\}$ and $\mathcal{F}'$ is the class of indicators of all half-spaces in $\mathbb{R}^n$. By Exercises 8.17 and 8.21, $\text{vc}(\mathcal{F}) \leq C(n+1) \leq Cn$. Thus, the VC law of large numbers (Theorem 8.3.15) gives

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\frac{1}{m}\sum_{i=1}^{m} f(X_i) - \mathbb{E}f(X_1)\right| \leq C\sqrt{\frac{n}{m}}.$$

By Markov's inequality and this expectation bound,

$$\mathbb{P}\left\{\sup_{f\in\mathcal{F}}\left|\frac{1}{m}\sum_{i=1}^{m} f(X_i) - \mathbb{E}f(X_1)\right| > 100C\sqrt{\frac{n}{m}}\right\} \leq \frac{1}{100}.$$

Thus, with probability at least 0.99, the required bound holds simultaneously for all $u, v \in S^{n-1}$.

8.27 (Random matrices with no moment assumptions) Most random matrix results assume something nice about the distribution—like subgaussian tails (Theorem 4.6.1) or at least a finite second moment (see Exercise 5.31). But now, you will use the impressive power of VC theory (specifically, Theorem 8.3.15) to prove that a tall random matrix is nicely invertible, even with no moment assumptions at all.

(a) Let $\varepsilon, \delta > 0$ and let X be a random vector in $\mathbb{R}^n$ that satisfies the following anti-concentration assumption:

$$\mathbb{P}\{|\langle X, u\rangle| \geq \varepsilon\} \geq \delta \qquad \forall u \in S^{n-1}. \tag{8.55}$$

Let A be an $m \times n$ random matrix whose rows are i.i.d. copies of the random vector X. Prove that if $m \geq C\delta^{-2}n$, then with probability at least 0.99, the smallest singular value of A is bounded away from zero:

$$s_n(A) \geq 0.99\varepsilon\sqrt{\delta m}.$$

(b) Demonstrate that the bound in part (a) is optimal. For every positive integers $m \geq n$ and positive reals ε and $\delta < 1/3$, find a random matrix A whose rows are i.i.d. copies of a random vector X that satisfies (8.55), and such that

$$\mathbb{E}s_n(A) \leq 1.01\varepsilon\sqrt{\delta m}.$$

Solution: (a) Write the rows of A as $X_1, \ldots, X_m$, by definition of $s_n(A)$, we have

$$s_n(A)^2 = \min_{u \in S^{n-1}} \sum_{i=1}^{m} \langle X_i, u\rangle^2.$$

Consider the function class

$$\mathcal{F} = \{\mathbb{1}\{|\langle x, u\rangle| \geq \varepsilon\} \mid u \in S^{n-1}\}.$$

By Exercise 8.17 and Exercise 8.21, there exists some constant C_1 such that $\mathrm{vc}(\mathcal{F}) \leq C_1 n$. By Theorem 8.3.15,

$$\mathbb{E} \sup_{u \in S^{n-1}} \left| \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{|\langle X_i, u\rangle| \geq \varepsilon\} - \mathbb{P}(|\langle X, u\rangle| \geq \varepsilon) \right| \leq C_2 \sqrt{\frac{n}{m}}.$$

By Markov inequality,

$$\mathbb{P}\left(\sup_{u \in S^{n-1}} \left| \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{|\langle X_i, u\rangle| \geq \varepsilon\} - \mathbb{P}(|\langle X, u\rangle| \geq \varepsilon) \right| \geq 100C_2 \sqrt{\frac{n}{m}} \right) \leq 0.01.$$

Therefore, with probability at least 0.99, we have for any $u \in S^{n-1}$,

$$\frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{|\langle X_i, u\rangle| \geq \varepsilon\} \geq \mathbb{P}(|\langle X, u\rangle| \geq \varepsilon) - 100C_2 \sqrt{\frac{n}{m}} \geq \delta - 100C_2 \sqrt{\frac{n}{m}}.$$

Choose the absolute constant $C \geq 10^8 C_2^2$. If $m \geq C\delta^{-2}n$, then $100C_2\sqrt{n/m} \leq 0.01\delta$. Hence, on

the same event,

$$s_n(A)^2 = \min_{u \in S^{n-1}} \sum_{i=1}^{m} \langle X_i, u \rangle^2 \geq \varepsilon^2 \min_{u \in S^{n-1}} \sum_{i=1}^{m} \mathbb{1}\{|\langle X_i, u \rangle| \geq \varepsilon\} \geq 0.99\varepsilon^2 \delta m.$$

Taking square roots gives $s_n(A) \geq \sqrt{0.99}\varepsilon\sqrt{\delta m} \geq 0.99\varepsilon\sqrt{\delta m}$.

(b) Let $g_2, \ldots, g_n$ be independent standard normal variables, and define

$$X = \begin{cases} 1.01\varepsilon e_1, & \text{with probability } \delta, \\ 20\varepsilon(0, g_2, \ldots, g_n), & \text{with probability } 1 - \delta. \end{cases}$$

Fix $u \in S^{n-1}$. If $|u_1| \geq 1/1.01$, the first value of X gives $|\langle X, u \rangle| \geq \varepsilon$, so (8.55) holds. Otherwise, $\sum_{j=2}^{n} u_j^2 = 1 - u_1^2 > 1 - 1/1.01^2 > 0.01$. Conditional on choosing the second case, the projection $\langle X, u \rangle$ is Gaussian with variance greater than $4\varepsilon^2$. Hence,

$$\mathbb{P}\{|\langle X, u \rangle| \geq \varepsilon\} \geq (1 - \delta)\mathbb{P}\{|g_2| \geq 1/2\} > \frac{1 - \delta}{2} > \delta,$$

where we used $\mathbb{P}\{|g_2| < 1/2\} \leq 1/\sqrt{2\pi} < 1/2$ and $\delta < 1/3$. Thus, (8.55) holds for every $u \in S^{n-1}$.

Let the rows of A be independent copies of X. Since $s_n(A) \leq \|Ae_1\|_2$, Jensen's inequality gives

$$\mathbb{E}s_n(A) \leq \mathbb{E}\|Ae_1\|_2 \leq \sqrt{m\mathbb{E}\langle X, e_1 \rangle^2} = 1.01\varepsilon\sqrt{\delta m},$$

as required.

8.28 (Glivenko–Cantelli = finite VC dimension) In Remark 8.3.19, we noted that if a Boolean class has finite VC dimension, it is uniform Glivenko–Cantelli. Show the converse: if a Boolean class is uniform Glivenko–Cantelli, then its VC dimension must be finite.

Solution: We show that any $\mathcal{F}$ with infinite VC dimension cannot be uniform Glivenko–Cantelli. For any $n > 0$, there exists Ω_0 that can be shattered by $\mathcal{F}$ with $|\Omega_0| = 2n$. Let μ be the uniform distribution on Ω_0, and let $X, X_1, \ldots, X_n$ be i.i.d. with distribution μ. The sampled $X_1, \ldots, X_n$ contain at most n distinct points, so at least n points of Ω_0 do not appear in the sample. Since Ω_0 is shattered by $\mathcal{F}$, there exists $f \in \mathcal{F}$ that equals one on the sampled points and zero on the remaining points of Ω_0. Thus, $\frac{1}{n}\sum_{i=1}^{n} f(X_i) = 1$, while $\mathbb{E}_\mu f(X) \leq 1/2$. Consequently, for

every n,

$$\sup_{\mu} \mathbb{P}_{\mu} \left\{ \sup_{f \in \mathcal{F}} \left| \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}_{\mu} f(X) \right| > \frac{1}{4} \right\} = 1,$$

contradicting the definition of a uniform Glivenko–Cantelli class.

8.29 (VC dimension of $(f-h)^2$) Let $\mathcal{F}$ be a class of Boolean functions on some set Ω, and let h be a Boolean function on Ω. Show that the class

$$\left\{ (f-h)^2 : f \in \mathcal{F} \right\}$$

has the same VC dimension as $\mathcal{F}$.

Solution: For any $x \in \Omega$, suppose $h(x) = 0$, then $(f-h)^2(x) = 0$ iff $f(x) = 0$ and $(f-h)^2(x) = 1$ iff $f(x) = 1$. Suppose $h(x) = 1$, then $(f-h)^2(x) = 0$ iff $f(x) = 1$ and $(f-h)^2(x) = 1$ iff $f(x) = 0$.

Consider any subset $\{x_1, \ldots, x_n\} \subset \Omega$, we say $(b_1, \ldots, b_n) \in \{0,1\}^n$ can be realized by some function class $\mathcal{G}$ if there exists $g \in \mathcal{G}$ such that $g(x_i) = b_i$ for $i = 1, \ldots, n$. WLOG, we assume $h(x_i) = 0$ for $i = 1, \ldots, d$ and $h(x_j) = 1$ for $j = d+1, \ldots, n$. We can observe that $(b_1, \ldots, b_d, b_{d+1}, \ldots, b_n)$ can be realized by $\mathcal{F}$ iff $(b_1, \ldots, b_d, 1-b_{d+1}, \ldots, 1-b_n)$ can be realized by $\{(f-h)^2 \colon f \in \mathcal{F}\}$. Therefore, $\{x_1, \ldots, x_n\}$ can be shattered by $\mathcal{F}$ iff it can be shattered by $\{(f-h)^2 \colon f \in \mathcal{F}\}$. We finish the proof.

8.30 (Learning with random labels) In our setup of a learning problem (Section 8.4), we assumed that labels $T(X)$ were fully determined by X, which rarely holds in real life. For instance, it is unrealistic to think a diagnosis $T(X) \in \{0,1\}$ is entirely determined by health data X. More often, a label Y is a random variable that is just correlated with X. Try to extend the learning theory (up to Theorem 8.4.5) to training data of the form

$$(X_i, Y_i), \qquad i = 1, \ldots, n,$$

where (X_i, Y_i) are independent copies of some random pair (X, Y) with $X \in \Omega$ and $Y \in \{0,1\}$.

Solution: Let $\mathcal{F}$ be a class of Boolean functions on Ω with finite VC dimension. Define the risk and the empirical risk by

$$R(f) = \mathbb{E}(f(X) - Y)^2, \qquad R_n(f) = \frac{1}{n} \sum_{i=1}^{n} (f(X_i) - Y_i)^2,$$

and let $f^* = \arg\min_{f\in\mathcal{F}} R(f)$ and $\hat{f}_n = \arg\min_{f\in\mathcal{F}} R_n(f)$.

View the data as random points in $\Omega \times \{0,1\}$, with the Boolean target $T(x,y) = y$, and extend each predictor f to $(x,y) \mapsto f(x)$. The VC dimension of the extended class equals $\mathrm{vc}(\mathcal{F})$: any shattered set of pairs has distinct first coordinates forming a shattered set in Ω, and any shattered set in Ω can be lifted by assigning $y = 0$ to every point. Thus, Theorem 8.4.5 applies and gives

$$\mathbb{E}R(\hat{f}_n) \le R(f^*) + C\sqrt{\frac{\mathrm{vc}(\mathcal{F})}{n}}.$$

8.31 (Learning a Lipschitz function) We proved the generalization bound (Theorem 8.4.5) only for Boolean classes, but we often have data with real labels. Say we want to learn a Lipschitz function $T : [0,1] \to [0,1]$ from its values on a random sample $X_1, \dots, X_n \sim \mathrm{Unif}[0,1]$. So, consider the hypothesis class

$$\mathcal{F} := \Big\{ f : [0,1] \to [0,1],\ \|f\|_{\mathrm{Lip}} \le 1 \Big\}$$

and assume that $T \in \mathcal{F}$. Show that the empirical risk minimization algorithm from Section 8.4.2 has a generalization bound similar to Theorem 8.4.5:

$$\mathbb{E}R(f_n^*) \le R(f^*) + \frac{C}{\sqrt{n}}.$$

Solution: Since we always have $R(f_n^*) - R(f^*) \le 2\sup_{f\in\mathcal{F}} |R_n(f) - R(f)|$, it suffices to bound $\mathbb{E}\sup_{f\in\mathcal{F}} |R_n(f) - R(f)|$. Write $X_f = R_n(f) - R(f)$ and $\ell_f = (f - T)^2$. For two hypotheses $f_1, f_2 \in \mathcal{F}$, we have $\ell_{f_1} - \ell_{f_2} = (f_1 - f_2)(f_1 + f_2 - 2T)$. Since f_1, f_2, T take values in $[0,1]$, it follows that $\|\ell_{f_1} - \ell_{f_2}\|_\infty \le 2\,\|f_1 - f_2\|_\infty$. Moreover,

$$X_{f_1} - X_{f_2} = \frac{1}{n}\sum_{i=1}^n Z_i, \qquad Z_i := (\ell_{f_1} - \ell_{f_2})(X_i) - \mathbb{E}(\ell_{f_1} - \ell_{f_2})(X).$$

The Z_i are independent, mean-zero random variables with $|Z_i| \le 4\,\|f_1 - f_2\|_\infty$. By Proposition 2.7.1 and this boundedness, there is an absolute constant $C_1 > 0$, independent of n, T, f_1, f_2, such that

$$\|X_{f_1} - X_{f_2}\|_{\psi_2} \le \frac{C_1}{\sqrt{n}}\,\|f_1 - f_2\|_\infty.$$

Fix an absolute constant $C_2 > 0$ for which the covering number bound from Exercise 8.9 reads $\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \varepsilon) \le C_2/\varepsilon$ for $0 < \varepsilon < 1$, and let $C_3 > 0$ be the absolute constant in Dudley

inequality. Since $T \in \mathcal{F}$, $X_T = 0$, and $\operatorname{diam}(\mathcal{F}, \|\cdot\|_\infty) \le 1$, Dudley inequality gives

$$\begin{aligned}\mathbb{E}\sup_{f\in\mathcal{F}} |X_f| &= \mathbb{E}\sup_{f\in\mathcal{F}} |X_f - X_T| \\ &\le \frac{C_1 C_3}{\sqrt{n}} \int_0^1 \sqrt{\log \mathcal{N}(\mathcal{F}, \|\cdot\|_\infty, \varepsilon)}\, d\varepsilon \\ &\le \frac{C_1 C_3 \sqrt{C_2}}{\sqrt{n}} \int_0^1 \varepsilon^{-1/2}\, d\varepsilon = \frac{C}{\sqrt{n}}.\end{aligned}$$

Combining this with the excess risk bound yields $\mathbb{E}R(f_n^*) \le R(f^*) + C/\sqrt{n}$.

8.32 (No learning with infinite VC dimension) Theorem 8.4.5 hints that to learn well, we need at least as many training points n as the VC dimension of the hypothesis class. Show that if $n < \frac{1}{2}\operatorname{vc}(\mathcal{F})$, then there exists a distribution on Ω where no algorithm can reliably learn the target function in $\mathcal{F}$. (Make this precise!)

Solution: Since $2n < \operatorname{vc}(\mathcal{F})$, there exists a set Ω_0 with cardinality $2n$ that can be shattered by $\mathcal{F}$. Let μ be the uniform distribution on Ω_0, and let $X, X_1, \dots, X_n$ be i.i.d. with distribution μ. The sampled $X_1, \dots, X_n$ contain at most n distinct points, so at least n points of Ω_0 do not appear in the sample. Since Ω_0 is shattered, choose one target in $\mathcal{F}$ for each of the 2^{2n} possible labelings of Ω_0, and denote this finite collection by $\mathcal{T}$. Fix any deterministic learning algorithm. For each candidate target $T \in \mathcal{T}$, let $\hat{f}_{n,T}$ denote its Boolean output when trained on $(X_i, T(X_i))_{i=1}^n$. Fix $X_1, \dots, X_n$ and an unobserved point $x \in \Omega_0$. Pair the targets in $\mathcal{T}$ whose labels on Ω_0 differ only at x. For each pair T, T', the training data are identical, so $\hat{f}_{n,T}(x) = \hat{f}_{n,T'}(x)$, whereas $T(x) \ne T'(x)$. Hence exactly one of these two predictions is wrong. Thus, the algorithm errs at x for exactly half of the targets. Summing over the at least n unobserved points and then taking expectation over $X_1, \dots, X_n$ gives, with $R_T(f) = \mu\{x : f(x) \ne T(x)\}$,

$$\frac{1}{2^{2n}} \sum_{T\in\mathcal{T}} \mathbb{E}R_T(\hat{f}_{n,T}) \ge \frac{n}{2n} \cdot \frac{1}{2} = \frac{1}{4}.$$

By averaging, there is a fixed target $T_* \in \mathcal{T}$, independent of the sample, for which $\mathbb{E}R_{T_*}(\hat{f}_{n,T_*}) \ge 1/4$. Since $0 \le R_{T_*}(\hat{f}_{n,T_*}) \le 1$,

$$\frac{1}{4} \le \mathbb{E}R_{T_*}(\hat{f}_{n,T_*}) \le \frac{1}{8} + \frac{7}{8}\mathbb{P}\{R_{T_*}(\hat{f}_{n,T_*}) \ge 1/8\}, \qquad \text{so} \qquad \mathbb{P}\{R_{T_*}(\hat{f}_{n,T_*}) \ge 1/8\} \ge \frac{1}{7}.$$

Here the probability is over the training sample. Hence, for this fixed μ, every deterministic

algorithm has some target in $\mathcal{F}$ with risk at least $1/8$ with probability at least $1/7$, even though $\inf_{f\in\mathcal{F}} R_{T_*}(f) = 0$.

8.33 (γ_2 functional is at least as good as Dudley) Show that γ_2 functional is bounded by Dudley integral:

$$\gamma_2(T,d) \le C\int_0^\infty \sqrt{\log \mathcal{N}(T,d,\varepsilon)}\,d\varepsilon.$$

Solution: Assume the Dudley integral is finite, since otherwise there is nothing to prove. Then $\mathcal{N}(T,d,\varepsilon) < \infty$ for every $\varepsilon > 0$. Define

$$\varepsilon_0 = \inf\{\varepsilon > 0 : \mathcal{N}(T,d,\varepsilon) = 1\}, \qquad \varepsilon_k = \inf\{\varepsilon > 0 : \mathcal{N}(T,d,\varepsilon) \le 2^{2^k}\}, \quad k \ge 1.$$

If $0 < \varepsilon < \varepsilon_0$, then $\mathcal{N}(T,d,\varepsilon) \ge 2$; if $0 < \varepsilon < \varepsilon_k$ for $k \ge 1$, then $\mathcal{N}(T,d,\varepsilon) > 2^{2^k}$. Thus, the rectangles

$$(0,\varepsilon_k) \times \big(\sqrt{\log 2}\,2^{(k-1)/2}, \sqrt{\log 2}\,2^{k/2}\big), \qquad k \ge 0,$$

lie below the graph of $\sqrt{\log \mathcal{N}(T,d,\varepsilon)}$. Comparing their total area with the integral gives

$$\sum_{k=0}^\infty 2^{k/2}\varepsilon_k \le C_1 \int_0^\infty \sqrt{\log \mathcal{N}(T,d,\varepsilon)}\,d\varepsilon,$$

for some absolute constant C_1. Fix $\delta > 0$. For each $k \ge 0$, choose an $(\varepsilon_k + \delta 2^{-k})$-net T_k^* with $|T_0^*| = 1$ and $|T_k^*| \le 2^{2^k}$ for $k \ge 1$. Then $\{T_k^*\}$ is admissible, so for some absolute constant C_2,

$$\begin{aligned}
\gamma_2(T,d) &\le \sup_{t\in T} \sum_{k=0}^\infty 2^{k/2} d(t, T_k^*) \\
&\le \sum_{k=0}^\infty 2^{k/2}(\varepsilon_k + \delta 2^{-k}) \\
&\le C_1 \int_0^\infty \sqrt{\log \mathcal{N}(T,d,\varepsilon)}\,d\varepsilon + C_2\delta.
\end{aligned}$$

Letting $\delta \to 0$ completes the proof.

8.34 (γ_2 functional can outperform Dudley) Sometimes the γ_2 functional (8.45) can be significantly smaller than the Dudley sum in (8.44). To see this, let's revisit example in Exercise 8.4:

$$T := \{0\} \cup \left\{ \frac{e_k}{\sqrt{1+\log k}},\ k = 1,\dots,n \right\}.$$

(a) Construct an admissible sequence (T_k) to check that the γ_2 functional of T (with respect to the Euclidean metric) is bounded:

$$\gamma_2(T,d) = \inf_{(T_k)} \sup_{t\in T} \sum_{k=0}^{\infty} 2^{k/2} d(t,T_k) \le C.$$

(b) Show that Dudley sum is unbounded:

$$\inf_{(T_k)} \sum_{k=0}^{\infty} 2^{k/2} \sup_{t\in T} d(t,T_k) \to \infty \qquad \text{as } n\to\infty.$$

Solution: (a) To simplify notations, we let $m_0 = 0$ and $m_k := \frac{e_k}{\sqrt{1+\log k}}$ for $k \ge 1$. Define $T_k^* = \{m_0, \dots, m_{2^{2^k}-1}\}$, then $\{T_k^*\}$ is an admissible sequence. Then we have

$$\gamma_2(T,d) \le \sup_j \sum_{k=0}^{\infty} 2^{k/2} d(m_j, T_k^*).$$

Note that $d(m_j, T_k^*) = 0$ for $j < 2^{2^k}$ and $d(m_j, T_k^*) \le 1/\sqrt{1+\log j}$ for $j \ge 2^{2^k}$. Thus for $j \ge 2$,

$$\sum_{k=0}^{\infty} 2^{k/2} d(m_j, T_k^*) \le \sum_{k=0}^{\lfloor \log_2 \log_2 j \rfloor} \frac{2^{k/2}}{\sqrt{1+\log j}} \le C_1 \frac{\sqrt{\log_2 j}}{\sqrt{1+\log j}} \le C_2,$$

for some absolute constants C_1, C_2. Hence $\gamma_2(T,d) \le C_2$.

(b) Fix any admissible sequence (T_k). For each k with $2^{2^k}+1 \le n$, since $|T_k| \le 2^{2^k}$, at least one of $m_1, \dots, m_{2^{2^k}+1}$ does not belong to T_k. Denote it by m_j. The nonzero points lie on different coordinate axes, so $\|m_j - m_i\|_2 \ge \|m_j\|_2$ for every $m_i \in T_k$. Hence

$$\begin{aligned} 2^{k/2} \sup_{t\in T} d(t,T_k) &\ge \frac{2^{k/2}}{\sqrt{1+\log j}} \\ &\ge \frac{2^{k/2}}{\sqrt{1+\log(2^{2^k}+1)}} \ge c_1, \end{aligned}$$

for some absolute constant $c_1 > 0$. Thus,

$$\inf_{(T_k)} \sum_{k=0}^{\infty} 2^{k/2} \sup_{t\in T} d(t,T_k) \ge c_2 \log\log n \to \infty.$$

8.35 (Generic chaining: a high-probability bound) Prove the result in Remark 8.5.4 by chaining with the first big leap:

$$t_0 \to \pi_\kappa(t) \to \pi_{\kappa+1}(t) \to \pi_{\kappa+2}(t) \to \cdots \to t,$$

where κ is chosen so that $u \asymp 2^{\kappa/2}$.

Solution: As before, we may assume that $K = 1$ and that T is finite, which makes $\gamma_2(T, d)$ finite. Let (T_k) be an admissible sequence of subsets of T such that

$$\sup_{t\in T} \sum_{k=0}^{\infty} 2^{k/2} d(t, T_k) \le 2\gamma_2(T, d).$$

Denote $T_0 = \{t_0\}$. Since T is finite, we may take $T_k = T$ for all sufficiently large k. For any $t \in T$, define $\pi_k(t)$ as the best approximation of t in T_k. We have $\pi_0(t) = t_0$ for any $t \in T$.

For every sample path of the process,

$$\sup_{t,s\in T} |X_t - X_s| \le \sup_{t\in T} |X_t - X_{t_0}| + \sup_{s\in T} |X_s - X_{t_0}| = 2 \sup_{t\in T} |X_t - X_{t_0}| .$$

It suffices to prove a high-probability bound for $\sup_{t\in T} |X_t - X_{t_0}|$. For $u \ge 1$, choose $\kappa = \lfloor 2 \log_2 u \rfloor$, so $2^\kappa \le u^2 < 2^{\kappa+1}$. We further write

$$\sup_{t\in T} |X_t - X_{t_0}| \le \sup_{t\in T} \left|X_t - X_{\pi_\kappa(t)}\right| + \sup_{t\in T} \left|X_{\pi_\kappa(t)} - X_{t_0}\right| .$$

We bound the two terms separately. For the first term, write

$$\left|X_t - X_{\pi_\kappa(t)}\right| \le \sum_{k=\kappa+1}^{\infty} \left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| .$$

The sub-gaussian assumption gives

$$\left\|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right\|_{\psi_2} \le d(\pi_k(t), \pi_{k-1}(t)).$$

Fix $t \in T$ and $k > \kappa$. For some absolute constant C_1, the event

$$\left|X_{\pi_k(t)} - X_{\pi_{k-1}(t)}\right| \le C_1 2^{k/2} d(\pi_k(t), \pi_{k-1}(t))$$

holds with probability at least $1 - 2\exp(-8 \cdot 2^k)$. Taking a union bound over at most $|T_k||T_{k-1}| \le 2^{2^{k+1}}$ pairs and all $k > \kappa$, the failure probability is at most

$$\sum_{k=\kappa+1}^{\infty} 2^{2^{k+1}} \cdot 2\exp(-8 \cdot 2^k) \le 2 \sum_{k=\kappa+1}^{\infty} \exp(-6 \cdot 2^k) \le \exp(-u^2).$$

On this event, the triangle inequality gives

$$\begin{aligned} \sup_{t \in T} \bigl|X_t - X_{\pi_\kappa(t)}\bigr| &\le C_1 \sup_{t \in T} \sum_{k=\kappa+1}^{\infty} 2^{k/2} \bigl(d(t, T_k) + d(t, T_{k-1})\bigr) \\ &\le C_2 \gamma_2(T, d) \end{aligned}$$

for some absolute constant C_2.

For the second term, $\|X_s - X_{t_0}\|_{\psi_2} \le d(s, t_0) \le \operatorname{diam}(T)$ for every $s \in T_\kappa$. Thus, for some absolute constant C_3, a union bound gives

$$\begin{aligned} \mathbb{P}\left\{ \sup_{s \in T_\kappa} |X_s - X_{t_0}| > C_3 u \operatorname{diam}(T) \right\} &\le 2|T_\kappa| \exp(-8u^2) \\ &\le 2\exp\bigl((\log 2) 2^\kappa - 8u^2\bigr) \le \exp(-u^2). \end{aligned}$$

Combining the two bounds, with probability at least $1 - 2\exp(-u^2)$,

$$\sup_{t,s \in T} |X_t - X_s| \le 2C_2 \gamma_2(T, d) + 2C_3 u \operatorname{diam}(T).$$

The case $0 \le u < 1$ follows from the bound at $u = 1$ and $\operatorname{diam}(T) \le 2\gamma_2(T, d)$. Thus, for some absolute constant C_4,

$$\sup_{t,s \in T} |X_t - X_s| \le C_4 \bigl[\gamma_2(T, d) + u \operatorname{diam}(T)\bigr]$$

with probability at least $1 - 2\exp(-u^2)$ for every $u \ge 0$.

8.36 (Empirical processes: a generic chaining bound) So far, we have bounded empirical processes for two kinds of function classes: Lipschitz (Theorem 8.2.3) and Boolean with finite VC dimension (Theorem 8.3.15). Now let's go general. Take any class $\mathcal{F}$ of real-valued functions on a domain Ω, and let $X, X_1, X_2, \dots, X_n$ be i.i.d. random points in Ω. Let d be a metric on $\mathcal{F}$ satisfying

$$\|f(X) - g(X)\|_{\psi_2} \le d(f, g) \qquad \text{for all } f, g \in \mathcal{F}.$$

(For instance, $d(f,g) = C\|f-g\|_{L^\infty}$ works.) Then show:

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\frac{1}{n}\sum_{i=1}^n f(X_i) - \mathbb{E}f(X)\right| \le \frac{C\gamma_2(\mathcal{F},d)}{\sqrt{n}}.$$

Solution: The statement needs an additional assumption. Indeed, if $\mathcal{F} = \{f\}$ with $f(x) = x$, X takes values ± 1 with equal probability, and $n = 1$, then $\gamma_2(\mathcal{F},d) = 0$ while the left-hand side equals 1. We therefore assume $0 \in \mathcal{F}$ below.

By centering, we have

$$\|(f(X) - \mathbb{E}f(X)) - (g(X) - \mathbb{E}g(X))\|_{\psi_2} \lesssim \|f(X) - g(X)\|_{\psi_2} \le d(f,g).$$

Define the process $Z_f := \frac{1}{\sqrt{n}}\sum_{i=1}^n (f(X_i) - \mathbb{E}f(X))$, then

$$\|Z_f - Z_g\|_{\psi_2}^2 \lesssim \frac{1}{n}\sum_{i=1}^n \|(f(X_i) - \mathbb{E}f(X)) - (g(X_i) - \mathbb{E}g(X))\|_{\psi_2}^2 \lesssim d(f,g)^2.$$

Thus $\|Z_f - Z_g\|_{\psi_2} \lesssim d(f,g)$. Since $0 \in \mathcal{F}$ and $Z_0 = 0$, Remark 8.5.3 gives

$$\mathbb{E}\sup_{f\in\mathcal{F}}|Z_f| \le \mathbb{E}\sup_{f,g\in\mathcal{F}}|Z_f - Z_g| \lesssim \gamma_2(\mathcal{F},d).$$

Dividing by $\sqrt{n}$ completes the proof.

8.37 (Talagrand comparison inequality: a geometric form) Let's explore some useful variants of Corollary 8.5.8. Let $(X_x)_{x\in T}$ be a random process (not necessarily mean-zero) on a subset $T \subset \mathbb{R}^n$ such that[4] $X_0 = 0$. Assume that

$$\|X_x - X_y\|_{\psi_2} \le K\|x-y\|_2 \qquad \text{for all } x,y \in T\cup\{0\}.$$

(a) (Expectation) Show that

$$\mathbb{E}\sup_{x\in T}|X_x| \le CK\gamma(T),$$

where $\gamma(T) = \mathbb{E}\sup_{x\in T}|\langle g,x\rangle|$ is the Gaussian complexity of T (see Section 7.5.3).

[4] If the set T does not contain the origin, simply define $X_0 := 0$.

(b) (High-probability bound) Show that for every $u \geq 0$,

$$\sup_{x \in T} |X_x| \leq CK\big(w(T) + u \cdot \operatorname{rad}(T)\big)$$

with probability at least $1 - 2e^{-u^2}$, where $\operatorname{rad}(T) = \sup_{x \in T} \|x\|_2$ is the radius of T.

(c) (Moments) Conclude that

$$\left(\mathbb{E} \sup_{x \in T} |X_x|^p\right)^{1/p} \leq C\sqrt{p}\, K\gamma(T).$$

Solution: (a) From then on, we assume $K = 1$, T is finite, and $0 \in T$. Since the process $(X_x)_{x \in T}$ has sub-gaussian increments and $X_0 = 0$, by Remark 8.5.3,

$$\mathbb{E} \sup_{x \in T} |X_x| \leq \mathbb{E} \sup_{x,y \in T} |X_x - X_y| \lesssim \gamma_2(T, d),$$

where $d(x, y) \coloneqq \|x - y\|_2$. Consider the canonical process $Y_x = \langle x, g \rangle$, where $g \sim N(0, I_n)$. We have $w(T) = \mathbb{E} \sup_{x \in T} Y_x$ and $\gamma(T) = \mathbb{E} \sup_{x \in T} |Y_x|$. Then by Theorem 8.5.5,

$$\gamma_2(T, d) \lesssim \mathbb{E} \sup_{x \in T} Y_x \leq \mathbb{E} \sup_{x \in T} |Y_x|\,.$$

We finish the proof.

(b) By Remark 8.5.4, there exists some constant C_1 such that for every $u \geq 0$, the event

$$\sup_{x,y \in T} |X_x - X_y| \leq C_1(\gamma_2(T, d) + u \operatorname{diam}(T))$$

happens with probability at least $1 - 2e^{-u^2}$. Since $X_0 = 0$, we always have $\sup_{x \in T} |X_x| \leq \sup_{x,y \in T} |X_x - X_y|$. By Theorem 8.5.5, there exists constant C_2 such that $\gamma_2(T, d) \leq C_2 w(T) \leq C_2 \gamma(T)$. Also,

$$\operatorname{diam}(T) = \sup_{x,y \in T} \|x - y\|_2 \leq \sup_{x,y \in T} (\|x - 0\|_2 + \|y - 0\|_2) = 2 \operatorname{rad}(T).$$

Combining these bounds, with probability at least $1 - 2e^{-u^2}$,

$$\sup_{x \in T} |X_x| \leq C_1 C_2 w(T) + 2C_1 u \operatorname{rad}(T) \leq C_3 \left(w(T) + u \operatorname{rad}(T)\right)$$

for some absolute constant C_3.

(c) Since

$$\sqrt{\frac{2}{\pi}}\,\mathrm{rad}(T) = \sup_{x\in T}\mathbb{E}|\langle g, x\rangle| \le \mathbb{E}\sup_{x\in T}|\langle g, x\rangle| = \gamma(T)$$

and $w(T) \le \gamma(T)$, part (b) gives, for every $u \ge 0$,

$$\mathbb{P}\left\{\sup_{x\in T}|X_x| - C_3\gamma(T) > C_3\sqrt{\frac{\pi}{2}}\,u\gamma(T)\right\} \le 2e^{-u^2}.$$

Also, $\sup_{x\in T}|X_x| - C_3\gamma(T) \ge -C_3\gamma(T)$. Hence, for some absolute constant C_4,

$$\left\|\sup_{x\in T}|X_x| - C_3\gamma(T)\right\|_{\psi_2} \le C_4\gamma(T).$$

Thus, for $p \ge 1$,

$$\begin{aligned}\left\|\sup_{x\in T}|X_x|\right\|_{L_p} &\le \left\|\sup_{x\in T}|X_x| - C_3\gamma(T)\right\|_{L_p} + \|C_3\gamma(T)\|_{L_p}\\ &\le C_5\sqrt{p}\gamma(T) + C_3\gamma(T)\\ &\le C_6\sqrt{p}\gamma(T)\end{aligned}$$

for some absolute constants C_5, C_6.

8.38 (Expected ℓ^p norm of a random vector) In Exercise 3.5, we gave optimal bounds on the expected ℓ^p norm of a random vector with independent subgaussian coordinates. Now, let's drop the independence assumption. For a subgaussian random vector X in $\mathbb{R}^N$ and $p \in [1, \infty]$, show that

$$\mathbb{E}\,\|X\|_p \le \begin{cases} CK\sqrt{p}\,N^{1/p}, & p \le \log N,\\ CK\sqrt{\log N}, & p \ge \log N,\end{cases}$$

where $K = \|X\|_{\psi_2}$.

Solution: WLOG, we assume $K = 1$. Duality gives

$$\|X\|_p = \sup_{t\in B_q^N(1)}\langle X, t\rangle,$$

where $B_q^N(1)$ is the unit ℓ^q ball in $\mathbb{R}^N$ with $\frac{1}{p} + \frac{1}{q} = 1$. By Remark 8.5.9,

$$\mathbb{E}\,\|X\|_p = \mathbb{E} \sup_{t \in B_q^N(1)} \langle X, t \rangle \lesssim w(B_q^N(1)).$$

The result follows by applying Exercise 7.17.

8.39 (Gaussian Chevet inequality) Let A be an $m \times n$ random matrix with i.i.d. $\mathcal{N}(0,1)$ entries. Let $T \subset \mathbb{R}^n$ and $S \subset \mathbb{R}^m$ be arbitrary bounded sets.

(a) Show that Theorem 8.6.1 holds with sharp constant 1:

$$\mathbb{E} \sup_{x \in T,\, y \in S} \langle Ax, y \rangle \le w(T)\,\mathrm{rad}(S) + w(S)\,\mathrm{rad}(T).$$

(b) Prove that the converse inequality:

$$\mathbb{E} \sup_{x \in T,\, y \in S} \langle Ax, y \rangle \ge c\big[w(T)\,\mathrm{rad}(S) + w(S)\,\mathrm{rad}(T)\big].$$

Solution: (a) We have not been able to prove this part. The direct increment comparison suggested by the hint fails for general bounded sets. Consider the centered Gaussian processes

$$X_{uv} = \langle Au, v \rangle, \qquad Y_{uv} = \langle g, u \rangle\,\mathrm{rad}(S) + \langle h, v \rangle\,\mathrm{rad}(T),$$

where $g \sim N(0, I_n)$ and $h \sim N(0, I_m)$ are independent. To apply the Sudakov–Fernique inequality (Theorem 7.2.8), we would need

$$\mathbb{E}(X_{uv} - X_{wz})^2 \le \mathbb{E}(Y_{uv} - Y_{wz})^2 \qquad \text{for all } u, w \in T,\ v, z \in S.$$

This condition fails in general. Take $m = n = 1$ and $T = S = \{1/2, 1\}$, so both radii equal 1, $X_{uv} = Auv$, and $Y_{uv} = gu + hv$. Then

$$\mathbb{E}\big(X_{1,1} - X_{1/2,1/2}\big)^2 = \left(1 - \frac{1}{4}\right)^2 = \frac{9}{16} > \frac{1}{2} = \mathbb{E}\big(Y_{1,1} - Y_{1/2,1/2}\big)^2.$$

Thus, Sudakov–Fernique cannot be applied directly to these two processes.

(b) For each fixed $x \in T$, we have $Ax \overset{d}{=} \|x\|_2 h$, where $h \sim N(0, I_m)$. Hence

$$\mathbb{E} \sup_{x\in T,\, y\in S} \langle Ax, y\rangle \geq \sup_{x\in T} \mathbb{E} \sup_{y\in S} \langle Ax, y\rangle = \sup_{x\in T} \|x\|_2 \, w(S) = \operatorname{rad}(T) w(S).$$

Similarly, fixing $y \in S$ and using $A^\top y \overset{d}{=} \|y\|_2 \, g$ gives

$$\mathbb{E} \sup_{x\in T,\, y\in S} \langle Ax, y\rangle \geq \operatorname{rad}(S) w(T).$$

Taking the average of these two lower bounds proves the claim with $c = 1/2$.

8.40 (Chevet inequality: a high-probability version) Under the assumptions of Theorem 8.6.1, prove a tail bound for $\sup_{x\in T,\, y\in S} \langle Ax, y\rangle$.

Solution: WLOG, assume $K = 1$. Write $r_T = \operatorname{rad}(T)$ and $r_S = \operatorname{rad}(S)$. Define

$$U \coloneqq \{(r_S x, r_T y) : x \in T,\ y \in S\} \subset \mathbb{R}^{n+m}, \qquad Z_{(r_S x, r_T y)} \coloneqq \langle Ax, y\rangle.$$

Following the proof of Theorem 8.6.1, for $z = (r_S x, r_T y)$ and $z' = (r_S x', r_T y')$, we have

$$\begin{aligned} \|Z_z - Z_{z'}\|_{\psi_2} &\leq C_0\big(r_S \left\|x - x'\right\|_2 + r_T \left\|y - y'\right\|_2\big) \\ &\leq \sqrt{2} C_0 \left\|z - z'\right\|_2. \end{aligned}$$

Set $Z_0 = 0$. For independent $g \sim N(0, I_n)$ and $h \sim N(0, I_m)$, we have

$$\begin{aligned} w(U) &= \mathbb{E} \sup_{x\in T,\, y\in S} \big(r_S \langle g, x\rangle + r_T \langle h, y\rangle\big) = r_S w(T) + r_T w(S), \\ \operatorname{rad}(U) &= \sup_{x\in T,\, y\in S} \sqrt{r_S^2 \|x\|_2^2 + r_T^2 \|y\|_2^2} = \sqrt{2} r_T r_S. \end{aligned}$$

Thus, Exercise 8.37(b) gives, for every $u \geq 0$, with probability at least $1 - 2e^{-u^2}$,

$$\sup_{x\in T,\, y\in S} |\langle Ax, y\rangle| \leq C\big[w(T)\operatorname{rad}(S) + w(S)\operatorname{rad}(T) + u \operatorname{rad}(T)\operatorname{rad}(S)\big]$$

for some absolute constant C, which also bounds the supremum without the absolute value.

8.41 (The $p \to q$ norm of a random matrix) Now we've got tools to handle all p, q—not just a few special cases like before (see Exercise 4.44)!

(a) Let A be an $m \times n$ random matrix with independent, mean-zero, subgaussian rows A_i, and take any $p, q \in [1, \infty]$. Prove that

$$\mathbb{E}\,\|A\|_{p\to q} \le CK\big[r(n,p)w(m,q) + r(m,q')w(n,p')\big],$$

where $K = \max_i \|A_i\|_{\psi_2}$ and

$$r(n,p) = \begin{cases} 1, & p \in [1,2], \\ n^{\frac{1}{2}-\frac{1}{p}}, & p \in [2,\infty], \end{cases} \qquad w(n,p) = \begin{cases} \sqrt{p}\, n^{1/p}, & p \in [1, \log n], \\ \sqrt{\log n}, & p \in [\log n, \infty]. \end{cases}$$

Here p' and q' are the usual Hölder conjugates of p and q as in (1.22).

(b) Show this bound is tight for Gaussian matrices: if A has i.i.d. $\mathcal{N}(0,1)$ entries, then

$$\mathbb{E}\,\|A\|_{p\to q} \ge c\big[r(n,p)w(m,q) + r(m,q')w(n,p')\big].$$

Solution: (a) Let $T = B_p^n(1)$ and $S = B_{q'}^m(1)$. Then

$$\|A\|_{p\to q} = \sup_{x\in T} \|Ax\|_q = \sup_{x\in T,\, y\in S} \langle Ax, y\rangle.$$

For $s \in [1,\infty]$, Exercise 1.17(a) gives

$$\mathrm{rad}(B_s^d(1)) = \sup_{\|x\|_s \le 1} \|x\|_2 = r(d,s).$$

Exercise 7.17 gives

$$w(B_s^d(1)) = \mathbb{E}\,\|g\|_{s'} \asymp w(d,s'), \qquad g \sim N(0, I_d).$$

Applying Theorem 8.6.1, we obtain

$$\begin{aligned} \mathbb{E}\,\|A\|_{p\to q} &\le CK\big[\mathrm{rad}(T)w(S) + \mathrm{rad}(S)w(T)\big] \\ &\lesssim K\big[r(n,p)w(m,q) + r(m,q')w(n,p')\big]. \end{aligned}$$

(b) If A has i.i.d. $N(0,1)$ entries, Exercise 8.39(b) applied to the same sets yields

$$\mathbb{E}\,\|A\|_{p\to q} \ge \frac{1}{2}\big[\mathrm{rad}(T)w(S) + \mathrm{rad}(S)w(T)\big]$$

$$\gtrsim r(n,p)w(m,q) + r(m,q')w(n,p'),$$

which proves the lower bound.